# Relations and Functions

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# Review

- A **relation** between two variables x and *y* is a set of ordered pairs
- An **ordered pair** consist of a x and *y*-coordinate
  - **A relation** may be viewed as *ordered pairs, mapping design, table, equation, or written in sentences*
- x-values are **inputs**, **domain**, **independent variable**
- *y*-values are **outputs**, **range**, **dependent variable**

# Example 1

$$\{(0,-5),(1,-4),(2,-3),(3,-2),(4,-1),(5,0)\}$$

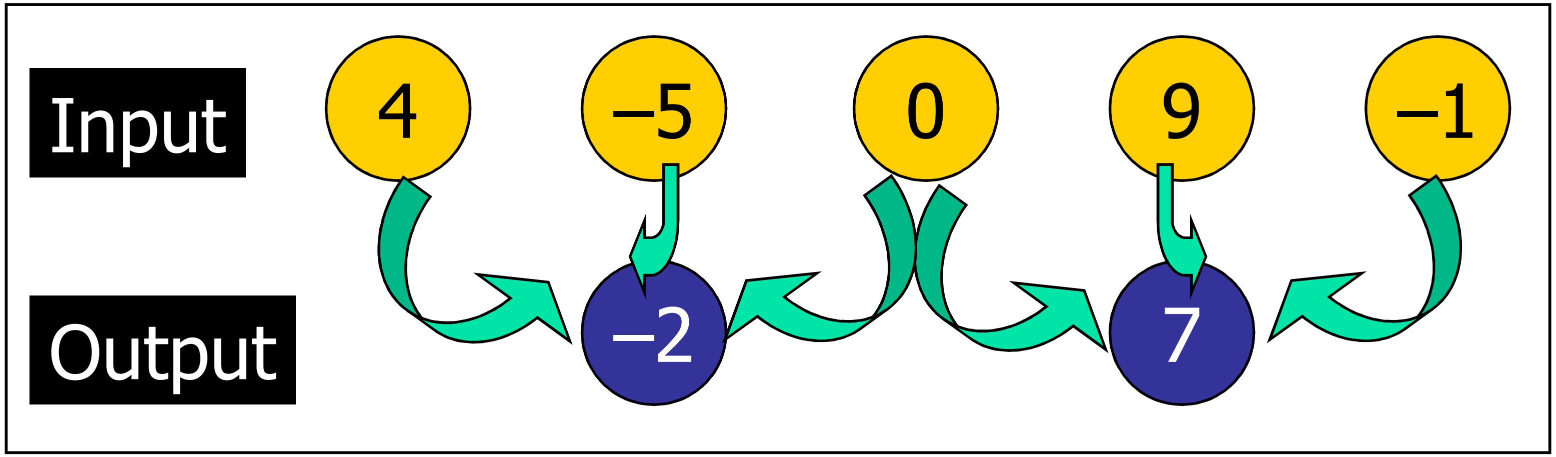
- What is the **domain**?

  **{0, 1, 2, 3, 4, 5}**

  What is the **range**?

  **{-5, -4, -3, -2, -1, 0}**

# Example 2




- What is the **domain**?

{4, -5, 0, 9, -1}

- What is the **range**?

{-2, 7}

## Is a relation a function?

# What is a **function**?

According to the textbook, "**a function is**...**a relation in which every input is paired with exactly one output**"

# Is a relation a function?

- Focus on the **x-coordinates**, when given a relation

If the set of ordered pairs have **different x-coordinates**,
it **IS A** function

If the set of ordered pairs have **same x-coordinates**,
it is **NOT** a function

- **Y-coordinates** have no bearing in determining functions

# Example 3

$$\{(0,-5),(1,-4),(2,-3),(3,-2),(4,-1),(5,0)\}$$

- ***Is this a function?***
  - ***Hint:*** *Look only at the **x-coordinates***

**YES**

# Example 4

$$\{(-1,-7),(1,0),(2,-3),(0,-8),(0,5),(-2,-1)\}$$

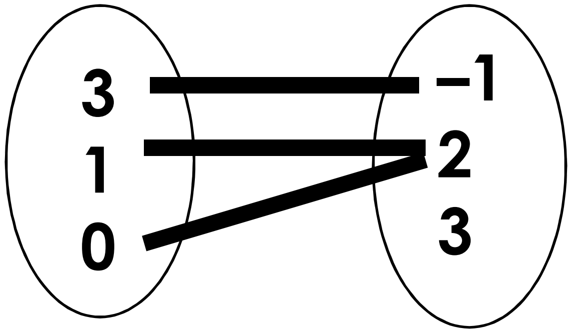
- ***Is this a function?***
  - ***Hint:*** *Look only at the* ***x-coordinates***

**NO**

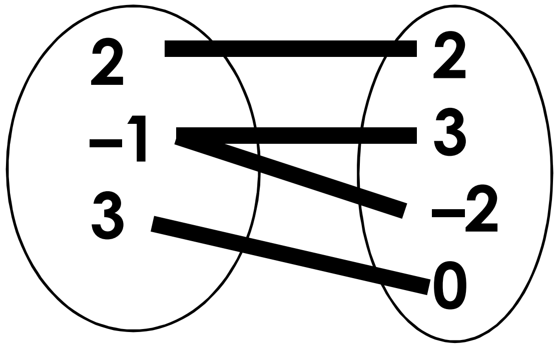
# Example 5

***Which mapping represents a function?***

Choice One




Choice Two

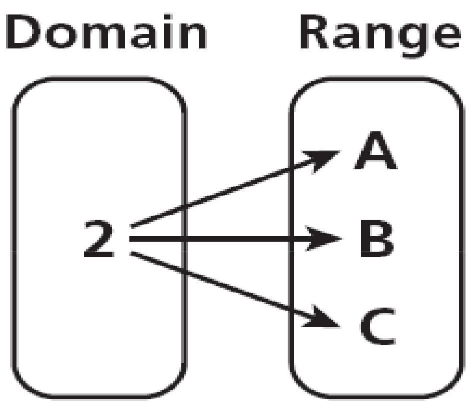



## Choice 1

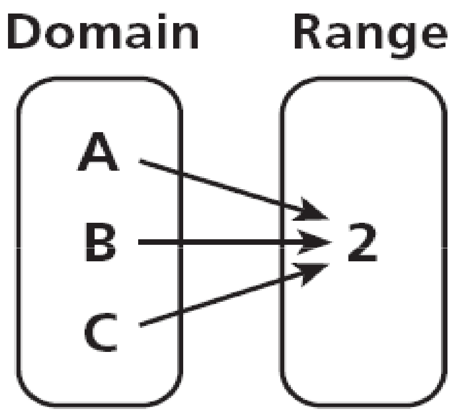
# Example 6

*Which mapping represents a function?*

A.




B.




**B**

# Example 7

*Which situation represents a function?*

a. The items in a store to their prices on a certain date

b. Types of fruits to their colors

*There is only one price for each different item on a certain date. The relation from items to price makes it a function.*

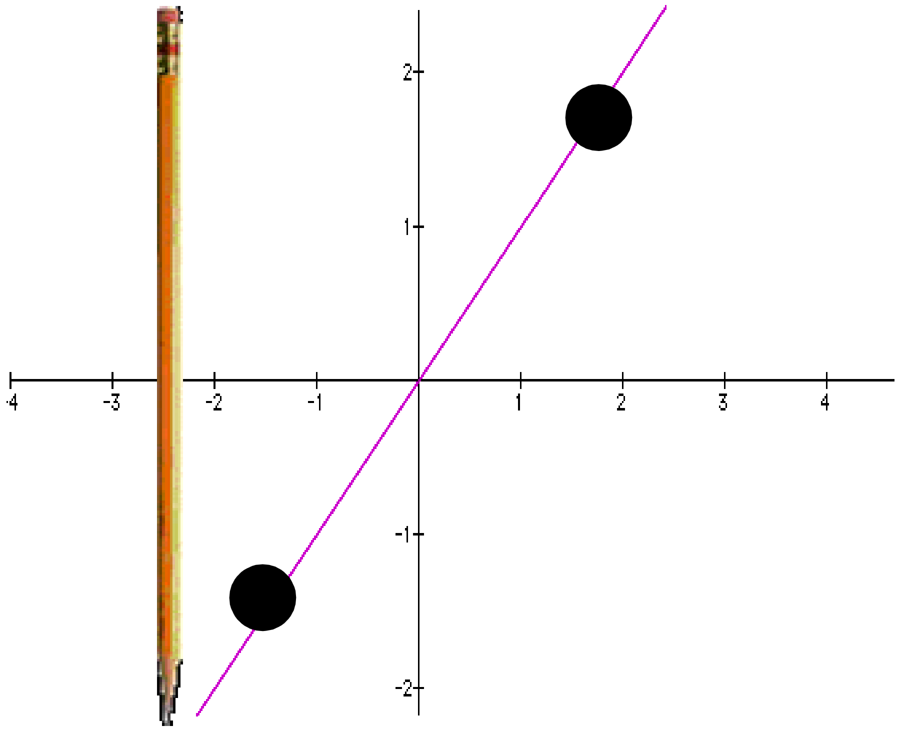
*A fruit, such as an apple, from the domain would be associated with more than one color, such as red and green. The relation from types of fruits to their colors is not a function.*

# Vertical Line Test

- **Vertical Line Test:** *a relation is a function if a vertical line drawn through its graph, passes through only one point.*

*AKA:* ***"The Pencil Test"***
*Take a pencil and move it from* ***left to right (–x to x)****; if it crosses more than one point, it is not a function*

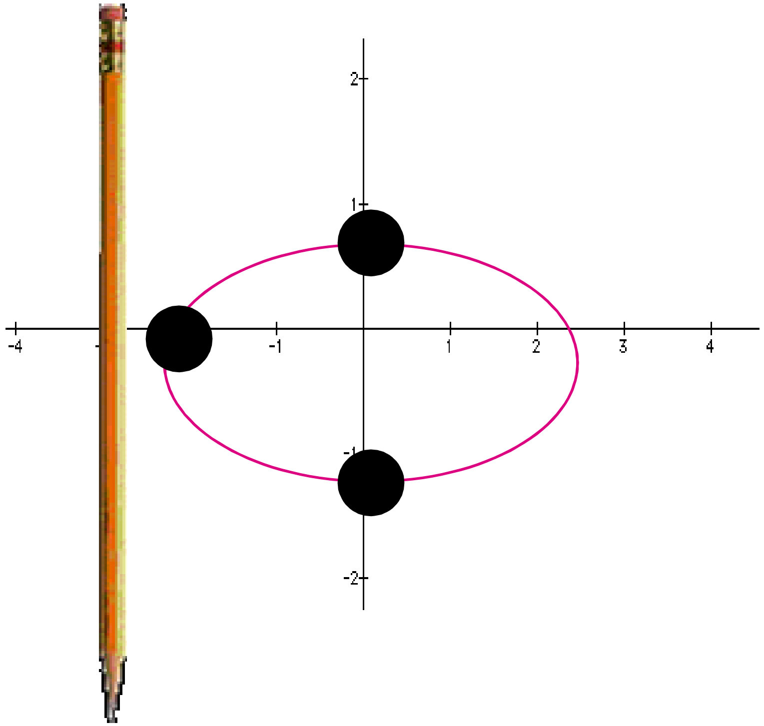
# Vertical Line Test




**Would this graph be a function?**

**YES**

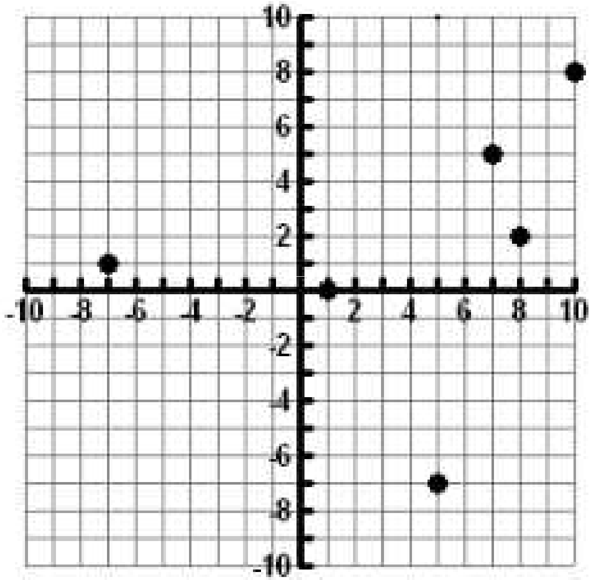
# Vertical Line Test




**Would this graph be a function?**

**NO**

Is the following function discrete or continuous? What is the Domain? What is the Range?

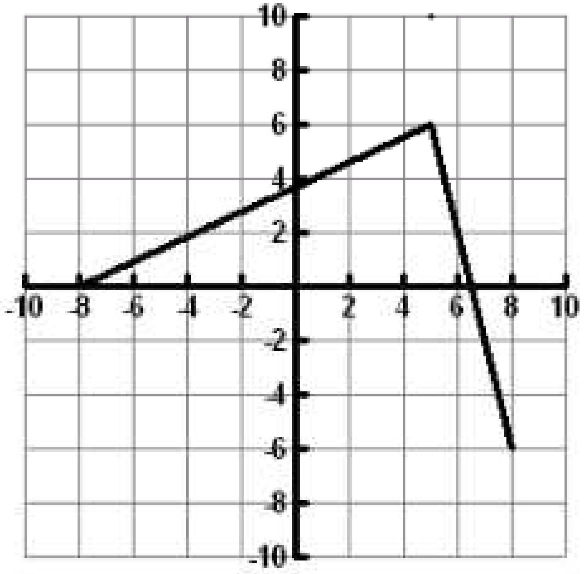



Type: Discrete

Domain: {-7, 1, 5, 7, 8, 10}

Range: {1, 0, -7, 5, 2, 8}

Is the following function discrete or continuous? What is the Domain? What is the Range?

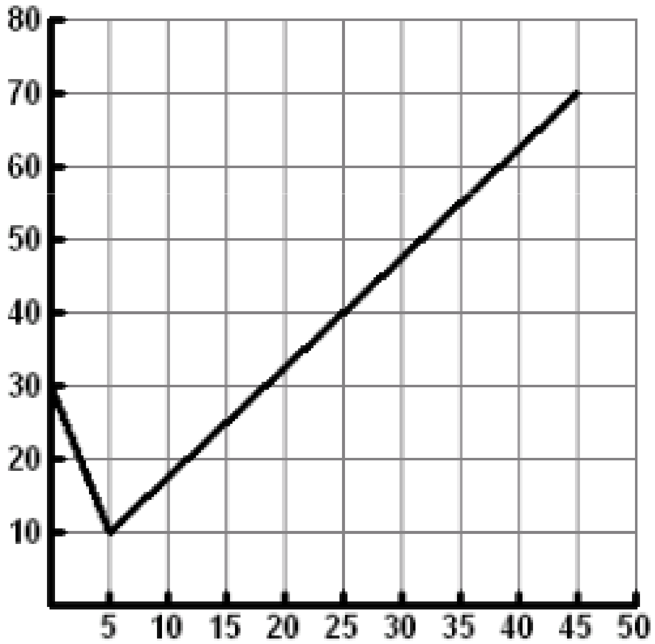



Type: continuous

Domain: $[-8,8]$

Range: $[-6,6]$

Is the following function discrete or continuous? What is the Domain? What is the Range?

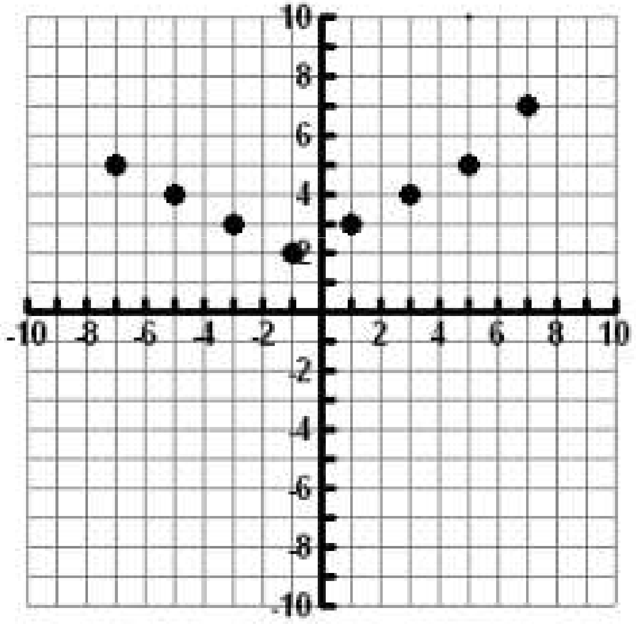



Type: continuous

Domain: [0,45]

Range: [10,70]

Is the following function discrete or continuous? What is the Domain? What is the Range?




Type: discrete

Domain: {-7, -5, -3, -1, 1, 3, 5, 7}

Range: {2, 3, 4, 5, 7}

# COMPLEX NUMBERS

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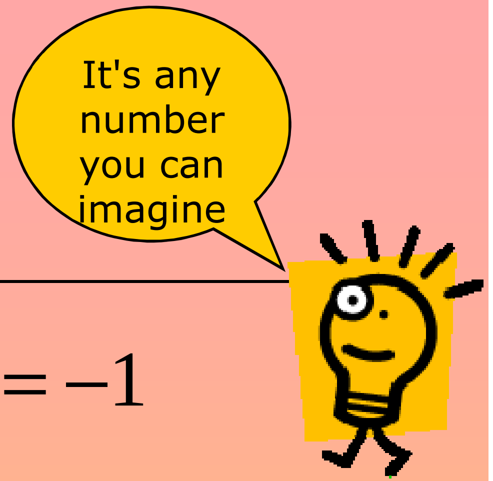
# The Imaginary Number ***i***




- By definition $\sqrt{-1} = i \iff i^2 = -1$

- Consider powers if *i*

$$i^2 = -1$$

$$i^3 = i^2 \cdot i = -i$$

$$i^4 = i^2 \cdot i^2 = -1 \cdot -1 = 1$$

$$i^5 = i^4 \cdot i = 1 \cdot i = i$$

...

# Using *i*

- Now we can handle quantities that occasionally show up in mathematical solutions

$$\sqrt{-a} = \sqrt{-1} \cdot \sqrt{a} = i\sqrt{a}$$

- What about

$$\sqrt{-49} \qquad \sqrt{-18}$$

# Complex Numbers

- Combine real numbers with imaginary numbers
  - a + b***i***

Real part

Imaginary part

- Examples

$$3 + 4i$$

$$-6 + \frac{3}{2}i$$

$$4.5 + i \cdot 2\sqrt{6}$$

# Try It Out

- Write these complex numbers in standard form   a + b***i***

$$9-\sqrt{-75}$$

$$\sqrt{-16}+7$$

$$5-\sqrt{-144}$$

$$-\sqrt{-100}$$

# Operations on Complex Numbers

- Complex numbers can be combined with
  - addition
  - subtraction
  - multiplication
  - division
- Consider $\left(-3+i\right)-\left(-8+2i\right)$

$$\left(9-12i\right)\cdot\left(7+15i\right)$$

$$\left(2-4i\right)+\left(4+3i\right)$$

# Operations on Complex Numbers

- Division technique
  - Multiply numerator and denominator by the conjugate of the denominator

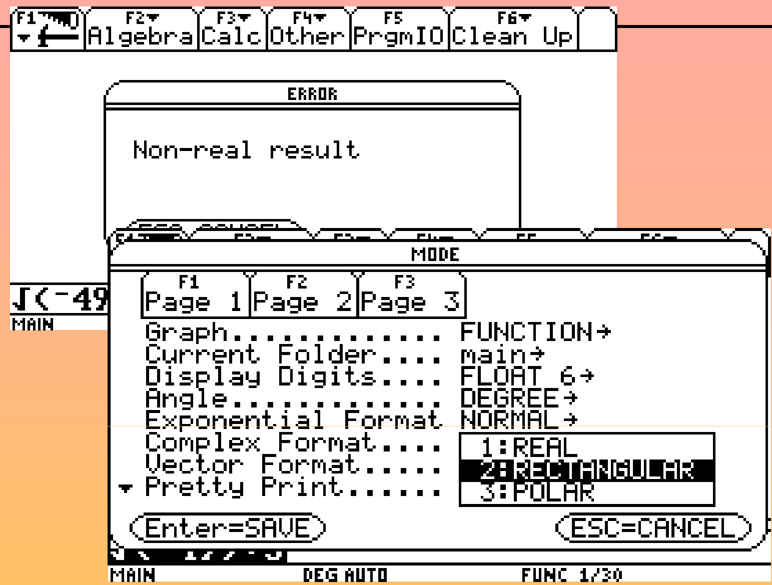
$$\frac{3i}{5-2i} = \frac{3i}{5-2i} \cdot \frac{5+2i}{5+2i}$$

$$= \frac{15i+6i^2}{25-4i^2}$$

$$= \frac{-6+15i}{29} = -\frac{6}{29} + \frac{15}{29}i$$

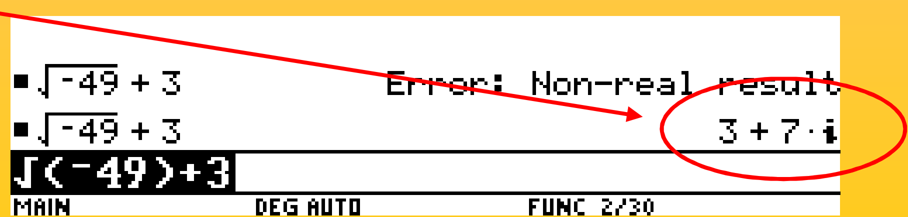
# Complex Numbers on the Calculator

- Possible result



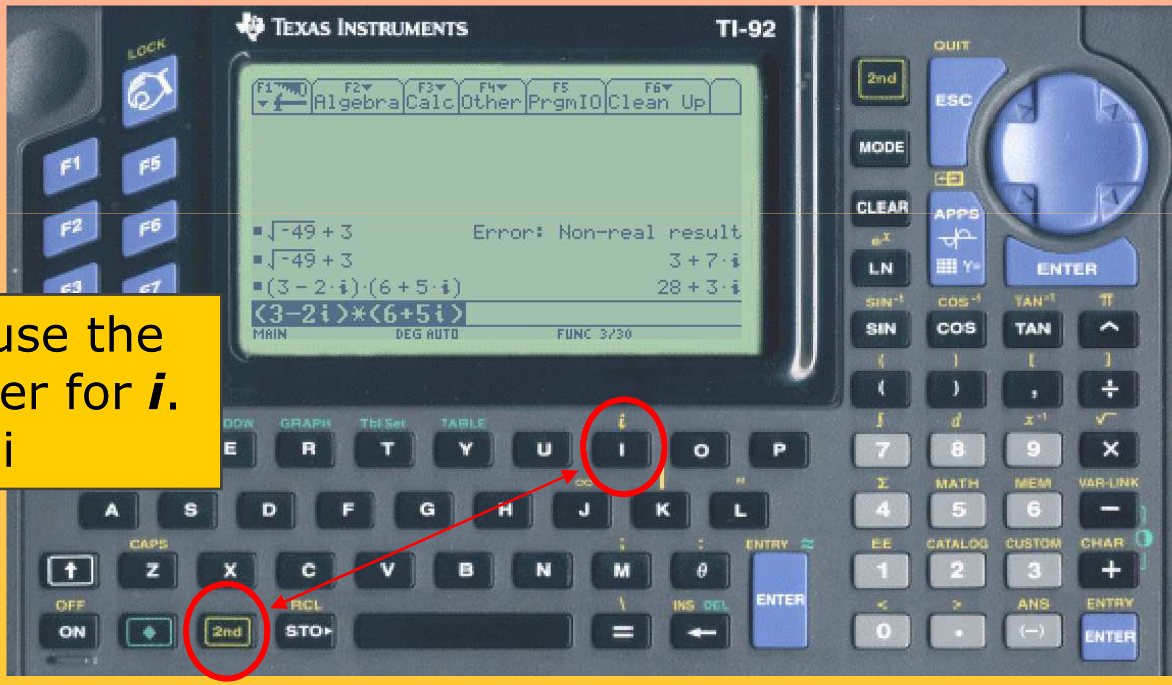

- Reset mode Complex format to <u>Rectangular</u>

- Now calculator does desired result



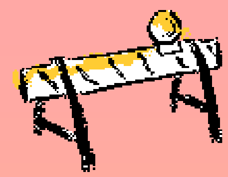
# Complex Numbers on the Calculator

- Operations with complex on calculator




Make sure to use the correct character for ***i***. Use 2nd-i

# Warning



- Consider $\sqrt{-16}\cdot\sqrt{-49}$
- It is tempting to combine them

$$\sqrt{-16\cdot -49} = \sqrt{+16\cdot 49} = 4\cdot 7 = 28$$


  - The multiplicative property of radicals only works for positive values under the radical sign
  - Instead use imaginary numbers

$$\sqrt{-16\cdot -49} = 4i\cdot 7i = 4\cdot 7\cdot i^2 = -28$$

# DIFFERENTIABILITY

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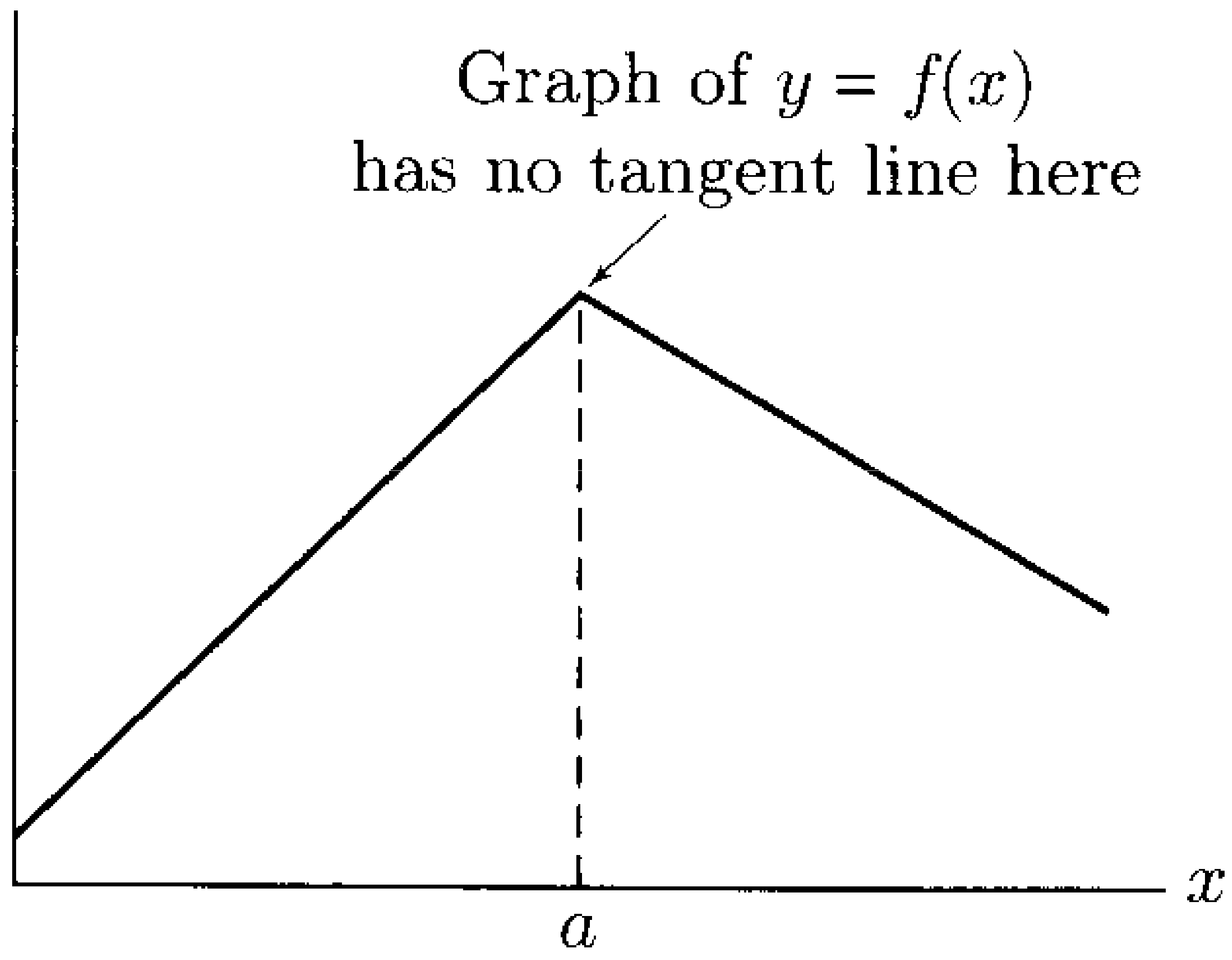
If a is a constant, we say that f(x) is differentiable at x = a if we can evaluate the following limit to determine $f^0(a)$.

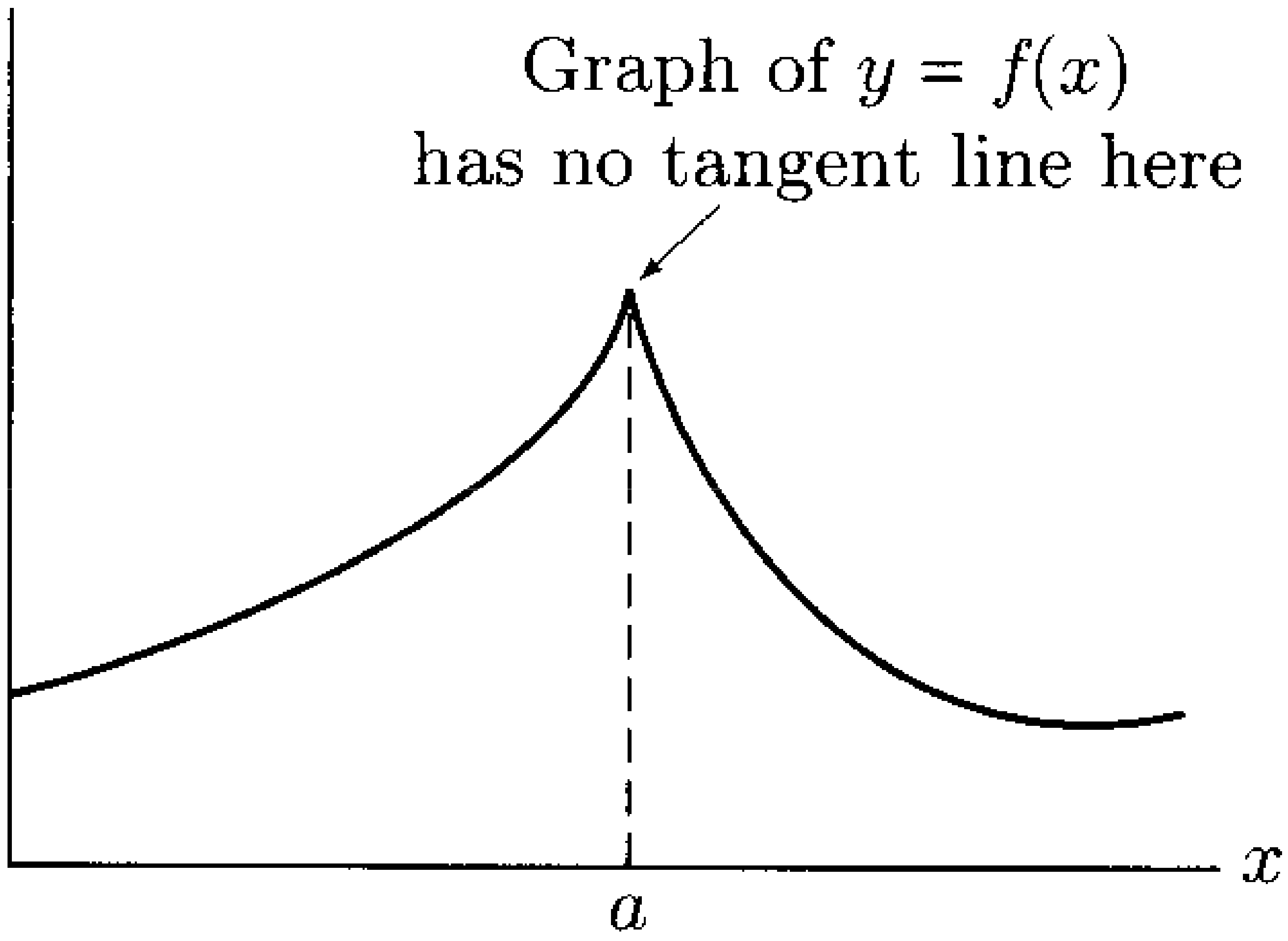
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

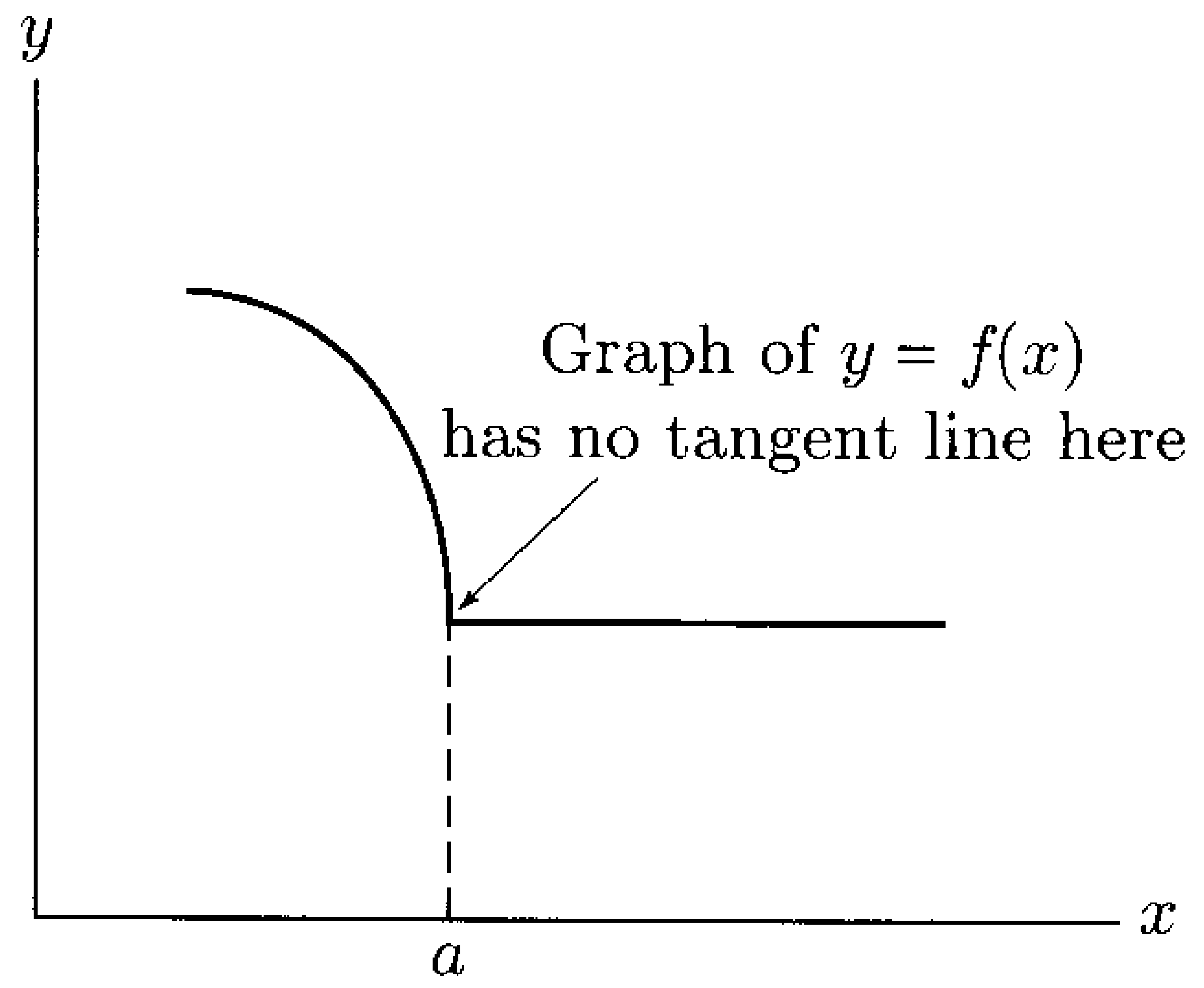
Conversely, if this limit does not exist, then f(x) is nondifferentiable at x = a.

There are many geometric representations of $f(x)$ for functions that are nondifferentiable at $x = a$.

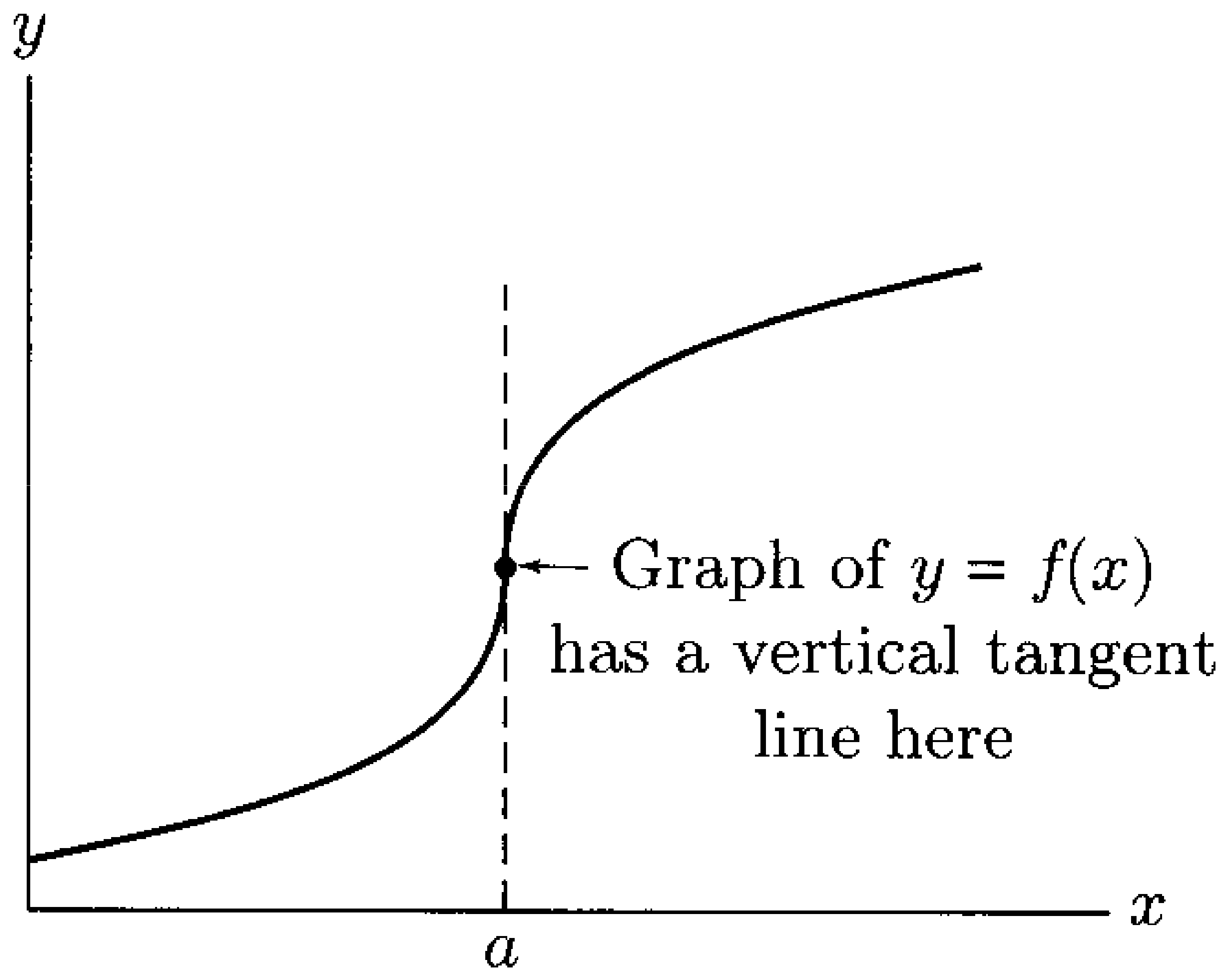
These can result if $f(x)$ has no tangent line at $x = a$, or if $f(x)$ has a vertical tangent line at $x = a$.


Graph of $y = f(x)$
has no tangent line here
$x$
$a$


Graph of y = f(x)
has no tangent line here
a
x



y
Graph of y = f(x)
has no tangent line here
x
a


y
x
a
Graph of y = f(x)
has a vertical tangent
line here

A railroad company charges $10 per mile to haul a boxcar up to 200 miles and $8 per mile for each mile exceeding 200. In addition, the railroad charges a $1000 handling charge per boxcar.

Graph the cost of sending a boxcar x miles.

If x is at most 200 miles, then the cost C(x) is given by:

$C(x) = 1000 + 10x$ dollars

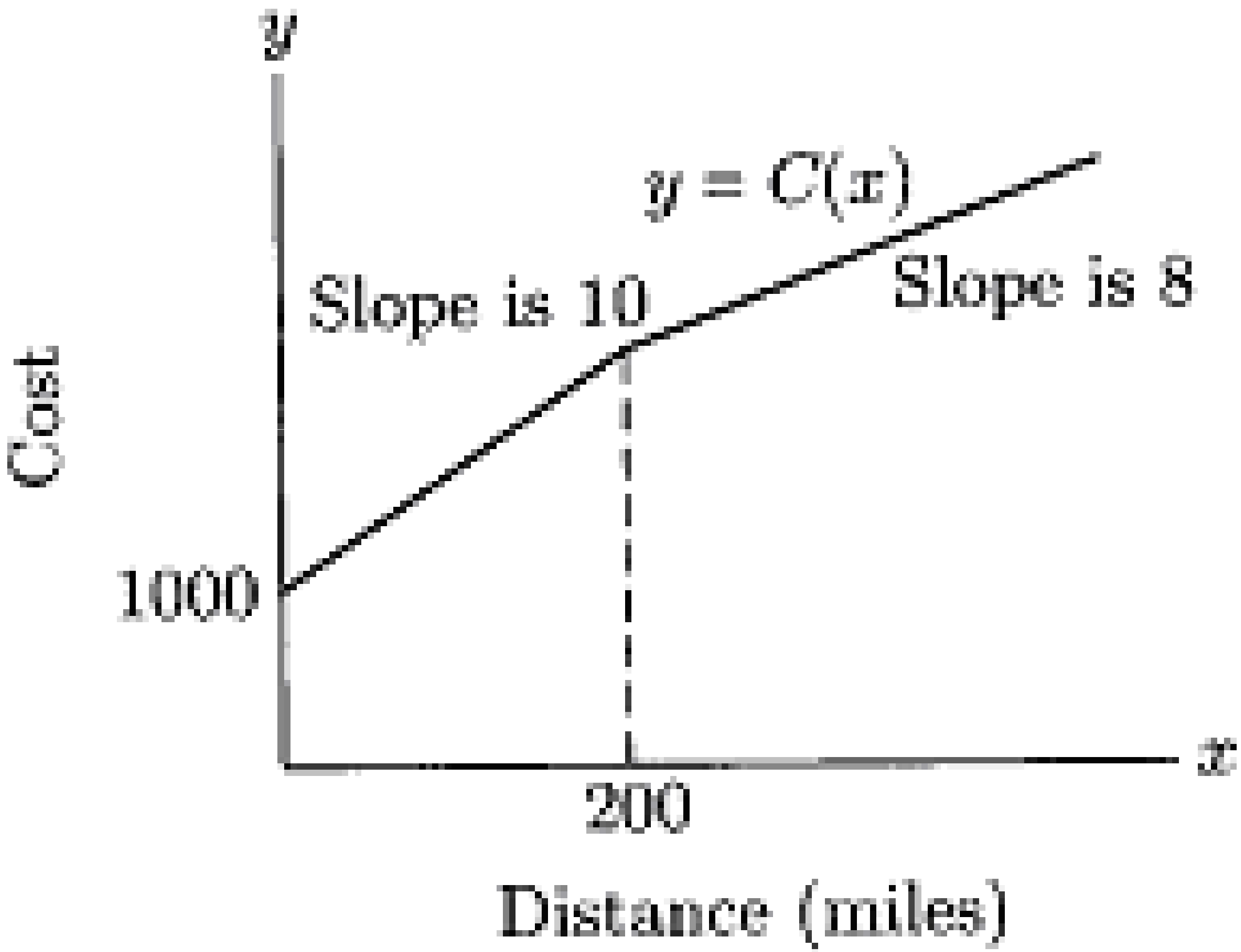
If x exceeds 200 miles, then the cost will be

$C(x) = 3000 + 8(x - 200) = 1400 + 8x$

So the function C(x) is given by

$$C(x) = \begin{cases} 1000 + 10x, & 0 < x \leq 200 \\ 1400 + 8x, & x > 200 \end{cases}$$

The graph of C(x) is



# THE INVERSE LAPLACE TRANSFORM

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# Inverse Laplace Transforms

Background:

To find the inverse Laplace transform we use transform pairs along with partial fraction expansion:

F(s) can be written as;

$$F(s) = \frac{P(s)}{Q(s)}$$

Where P(s) & Q(s) are polynomials in the Laplace variable, s. We assume the order of Q(s) $\geq$ P(s), in order to be in proper form. If F(s) is not in proper form we use long division and divide Q(s) into P(s) until we get a remaining ratio of polynomials that are in proper form.

# Inverse Laplace Transforms

Background:

**There are three cases to consider in doing the partial fraction expansion of F(s).**

**Case 1: F(s) has all non repeated simple roots.**

$$F(s) = \frac{k_1}{s+p_1} + \frac{k_2}{s+p_2} + \ldots + \frac{k_n}{s+p_n}$$

**Case 2: F(s) has complex poles:**

$$F(s) = \frac{P_1(s)}{Q_1(s)(s+\alpha-j\beta)(s+\alpha+j\beta)} = \frac{k_1}{s+\alpha-j\beta} + \frac{k_1^*}{s+\alpha+j\beta} + \ldots + \text{(expanded)}$$

**Case 3: F(s) has repeated poles.**

$$F(s) = \frac{P_1(s)}{Q_1(s)(s+p_1)^r} = \frac{k_{11}}{s+p_1} + \frac{k_{12}}{(s+p_1)^2} + \ldots + \frac{k_{1r}}{(s+p_1)^r} + \ldots + \frac{P_1(s)}{Q_1(s)}\text{(expanded)}$$

# Inverse Laplace Transforms

## Case 1: Illustration:

**Given:**

$$F(s) = \frac{4(s+2)}{(s+1)(s+4)(s+10)} = \frac{A_1}{(s+1)} + \frac{A_2}{(s+4)} + \frac{A_3}{(s+10)}$$

**Find $A_1$, $A_2$, $A_3$ from Heavyside**

$$A_1 = \frac{(s+1)4(s+2)}{(s+1)(s+4)(s+10)}\Big|_{s=-1} = 4/27 \qquad A_2 = \frac{(s+4)4(s+2)}{(s+1)(s+4)(s+10)}\Big|_{s=-4} = 4/9$$

$$A_3 = \frac{(s+10)4(s+2)}{(s+1)(s+4)(s+10)}\Big|_{s=-10} = -16/27$$

$$f(t) = \left[(4/27)e^{-t} + (4/9)e^{-4t} + (-16/27)e^{-10t}\right]u(t)$$

# Inverse Laplace Transforms

**<u>Case 3:</u> Repeated roots.**

**When we have repeated roots we find the coefficients of the terms as follows:**

$$k_{1r-1} = \frac{d}{ds}\left[(s+p_1)^r F(s)\right]\Big|_{s=-p_1}$$

$$k_{1r-2} = \frac{d^2}{2!\,ds^2}\left[(s+p_1)^r F(s)\right]\Big|_{s=-p_1}$$

$$k_{1j} = \frac{d^{r-j}}{(r-j)!\,ds^{r-j}}\left[(s+p_1)^r F(s)\right]\Big|_{s=-p_1}$$

# Inverse Laplace Transforms

**Case 3:** **Repeated roots. Example**

$$F(s) = \frac{(s+1)}{s(s+3)^2} = \frac{A_1}{s} + \frac{K_1}{(s+3)} + \frac{K_2}{(s+3)^2}$$

$A_1 =$

$K_1 =$

$K_2 =$

$$f(t) = [[\underline{\quad ? \quad}] + [\underline{\quad ? \quad}]e^{-3t} + [\underline{\quad ? \quad}]te^{-3t}]u(t)$$

# Inverse Laplace Transforms

**Case 2: Complex Roots:** **F(s) is of the form;**

$$F(s)=\frac{P_1(s)}{Q_1(s)(s+\alpha-j\beta)(s+\alpha+j\beta)}=\frac{K_1}{s+\alpha-j\beta}+\frac{K_1^*}{s+\alpha+j\beta}+\ldots+$$

**$K_1$ is given by,**

$$K_1=\frac{(s+\alpha-j\beta)P_1(s)}{Q_1(s)\,(s+\alpha-j\beta)(s+\alpha+j\beta)}\Big|_{s=-\alpha-j\beta}$$

$$K_1=|K_1|\angle\theta=|K_1|e^{j\theta}$$

# Inverse Laplace Transforms

**Case 2: Complex Roots**:

$$\frac{K_1}{s+\alpha-j\beta}+\frac{K_1^*}{s+\alpha+j\beta} = \frac{|K_1|e^{j\theta}}{s+\alpha-j\beta}+\frac{|K_1e^{-j\theta}}{s+\alpha+j\beta}$$

$$L^{-1}\left[\frac{|K_1|e^{j\theta}}{s+\alpha-j\beta}+\frac{|K_1|e^{-j\theta}}{s+\alpha+j\beta}\right]=|K|_1\left[e^{j\theta}e^{-\alpha t}e^{j\beta t}+e^{-j\beta}e^{-\alpha t}e^{-j\beta t}\right]$$

$$|K|_1\left[e^{j\theta}e^{-\alpha t}e^{j\beta t}+e^{-j\beta}e^{-\alpha t}e^{-j\beta t}\right]=2|K_1|e^{-at}\left[\frac{e^{j(\beta t+\theta)}+e^{j(\beta t+\theta)}}{2}\right]$$

# Inverse Laplace Transforms

**Case 2: Complex Roots:**

**Therefore:**

$$L^{-1}\left[\frac{|K_1|e^{j\theta}}{s+\alpha-j\beta}+\frac{|K_1e^{-j\theta}}{s+\alpha+j\beta}\right] = 2|K_1|e^{-\alpha t}\left[\cos(\beta t+\theta)\right]$$

**You should put this in your memory:**

# Inverse Laplace Transforms

## Complex Roots: An Example.

## For the given F(s) find f(t)

$$F(s) = \frac{(s+1)}{s(s^2+4s+5)} = \frac{(s+1)}{s(s+2-j)(s+2+j)}$$

$$F(s) = \frac{A}{s} + \frac{K_1}{s+2-j} + \frac{K_1^*}{s+2+j}$$

$$A = \frac{(s+1)}{(s^2+4s+5)}\bigg|_{s=0} = \frac{1}{5}$$

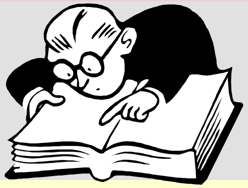
$$K_1 = \frac{(s+1)}{s(s+2+j)}\bigg|_{s=-2+j} = \frac{-2+j+1}{(-2+j)(2j)} = 0.32\angle -108^o$$

# Inverse Laplace Transforms

**Complex Roots: An Example.** (continued)

**We then have;**

$$F(s) = \frac{0.2}{s} + \frac{0.32\angle -108^o}{s+2-j} + \frac{0.32\angle +108^o}{s+2+j}$$



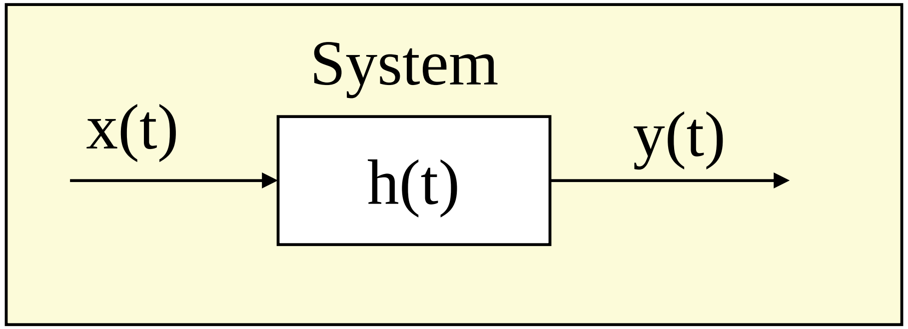
**Recalling the form of the inverse for complex roots;**

$$f(t) = \left[\, 0.2 + \; 0.64e^{-2t}\cos(t-108^o \;\right]u(t)$$

# Inverse Laplace Transforms

**Convolution Integral:**

Consider that we have the following situation.




x(t) is the input to the system.
h(t) is the impulse response of the system.
y(t) is the output of the system.

We will look at how the above is related in the time domain and in the Laplace transform.

# Inverse Laplace Transforms

**Convolution Integral:**

In the time domain we can write the following:

$$y(t) = x(t) \oplus h(t) = \int_{\tau=0}^{\tau=t} x(t-\tau)h(\tau)d\tau = \int_{\tau=0}^{\tau=t} h(t-\tau)x(\tau)d\tau$$

**In this case x(t) and h(t) are said to be convolved and the integral on the right is called the convolution integral.**

**It can be shown that,**

$$L\left[x(t) \oplus h(t)\right] = Y(s) = X(s)H\left(s\right)$$

**This is very important**

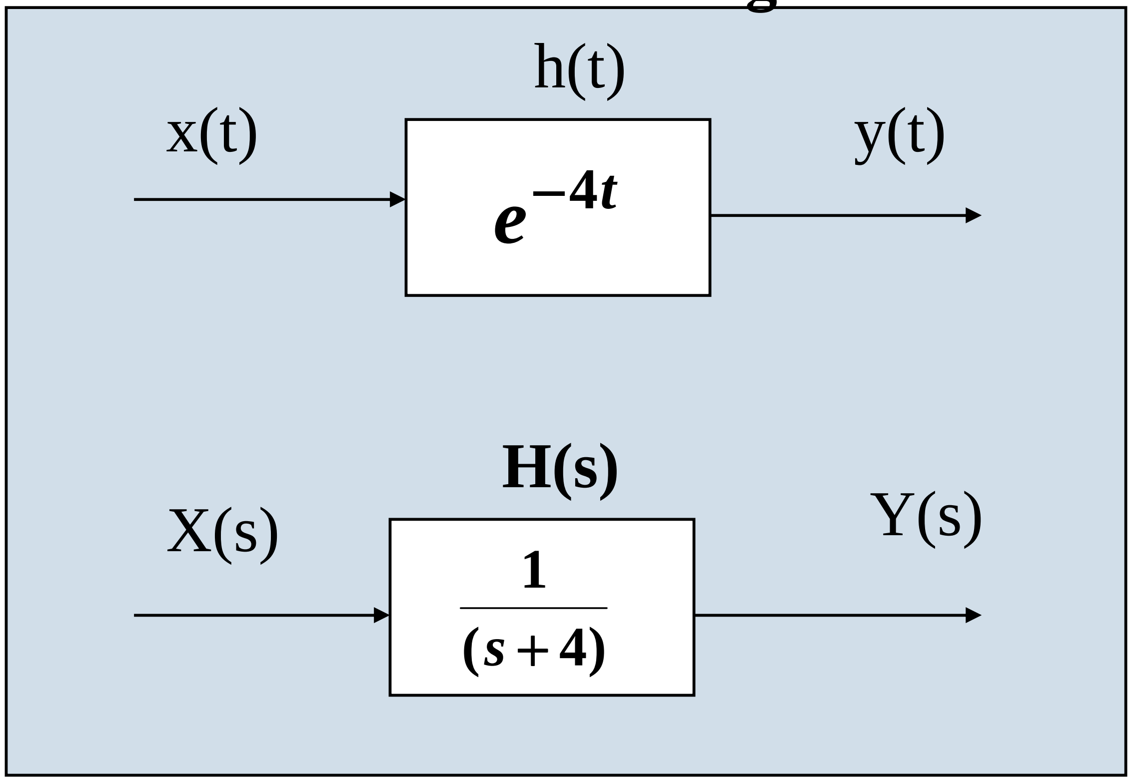
* note

# Inverse Laplace Transforms

**Convolution Integral:**

**Through an example let us see how the convolution integral and the Laplace transform are related.**

**We now think of the following situation:**



# Inverse Laplace Transforms

**Convolution Integral:**

**From the previous diagram we note the following:**

$$X(s) = L[x(t)]; \quad Y(s) = L[y(t)]; \quad H(s) = L[h(t)]$$

**h(t) is called the system impulse response for the following reason.**

$$Y(s) = X(s)H(s) \qquad \text{Eq A}$$

**If the input x(t) is a unit impulse, δ(t), the L(x(t)) = X(s) = 1. Since x(t) is an impulse, we say that y(t) is the impulse response. From Eq A, if X(s) = 1, then Y(s) = H(s). Since,**

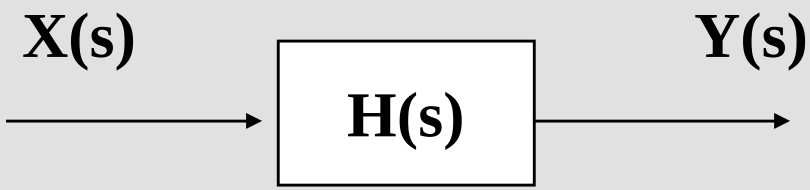
$$L^{-1}[Y(s)] = y(t) = impulse\ \ response = L^{-1}[H(s)] = h(t)$$

$$So,\ h(t) = system\ \ impulse\ \ response.$$

# Inverse Laplace Transforms

**Convolution Integral:**

**A really important thing here is that anytime you are given a system diagram as follows,**



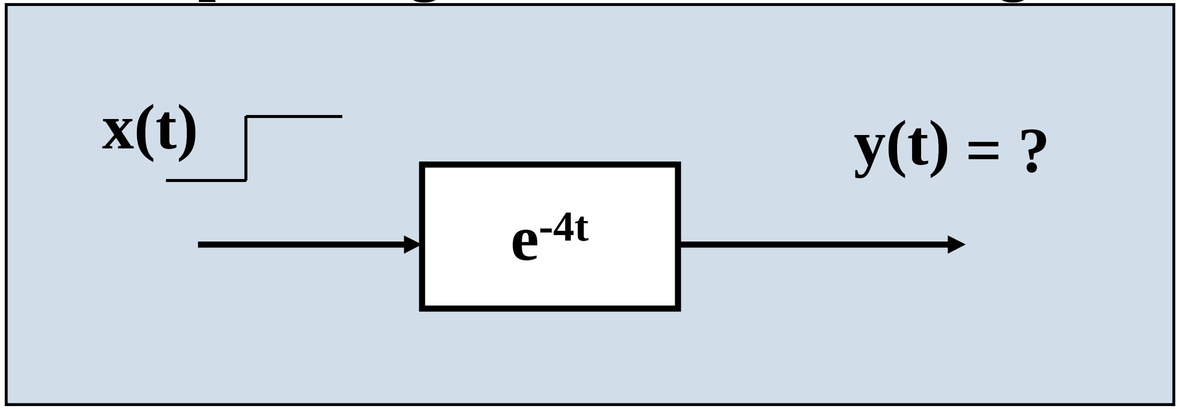

**the inverse Laplace transform of H(s) is the system's impulse response.**

**<u>This is important !!</u>**

# Inverse Laplace Transforms

## Convolution Integral:

**Example using the convolution integral.**




$$y(t)=\int_{-\infty}^{+\infty} e^{-4(t-\tau)}u(\tau)d\tau=\int_{0}^{t} e^{-4(t-\tau)}d\tau=e^{-4t}\int_{0}^{t} e^{4\tau}d\tau$$

$$y(t)=e^{-4t}\int_{0}^{t} e^{4\tau}d\tau \;=e^{-4t}\frac{1}{4}e^{4\tau}\Big|_{\tau=0}^{\tau=t}=\left[\frac{1}{4}-\frac{1}{4}e^{-4t}\right]u(t)$$

# Inverse Laplace Transforms

**Convolution Integral:**

**Same example but using Laplace.**

x(t) = u(t) $\longrightarrow$ $X(s) = \frac{1}{s}$

h(t) = $e^{-4t}$u(t) $\longrightarrow$ $H(s) = \frac{1}{s+4}$

$$Y(s) = \frac{1}{s(s+4)} = \frac{A}{s} + \frac{B}{s+4} = \frac{1/4}{s} - \frac{1/4}{s+4}$$

$$y(t) = \frac{1}{4}\left[1 - e^{-4t}\right]u(t)$$

# Inverse Laplace Transforms

**Convolution Integral:**

**Practice problems:**

$$(a)\ If\ \ X(s)=\frac{2}{s}\ and\ Y(s)=\frac{3}{(s+2)},\ what\ is\ h(t)?$$

$$h(t)=1.5\left[\delta(t)-2e^{-2t}u(t)\right]$$

$$(b)\ \ If\ \ x(t)=u(t)\ and\ \ y(t)=te^{-6t}u(t),\ \ find\ \ h(t).$$

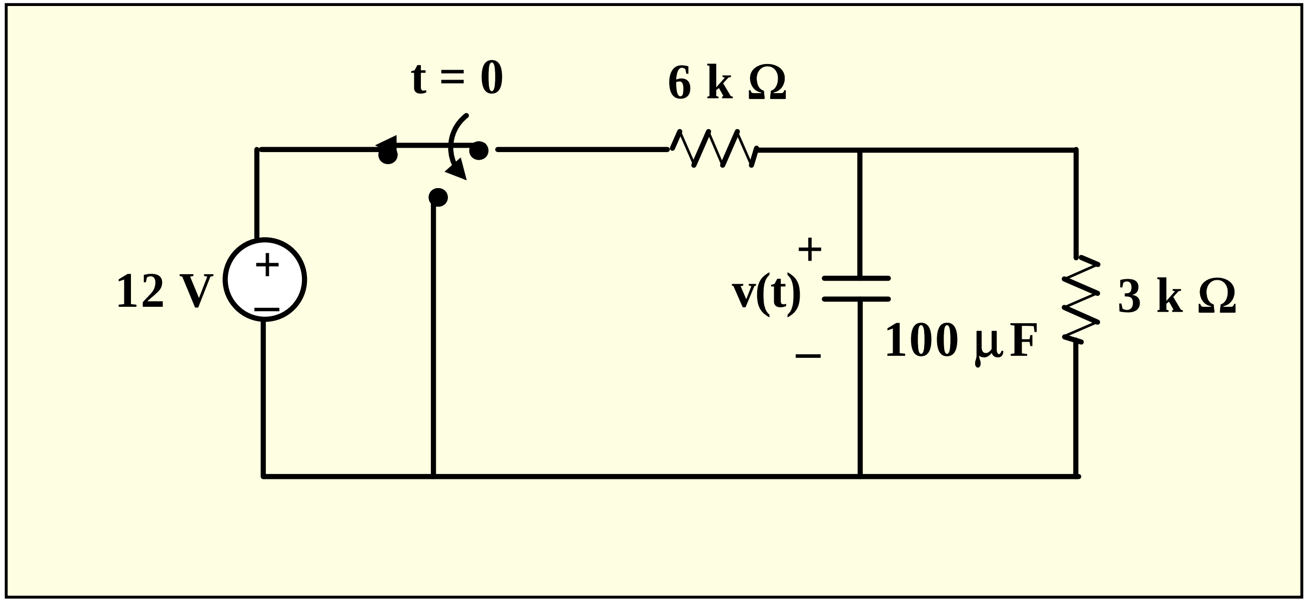
$$(c)\ If\ \ x(t)=tu(t)\ and\ \ H(s)=\frac{2}{(s+4)^2},\ \ find\ \ y(t).$$

**Answers given on note page**

# Inverse Laplace Transforms

**Circuit theory problem:**

**You are given the circuit shown below.**

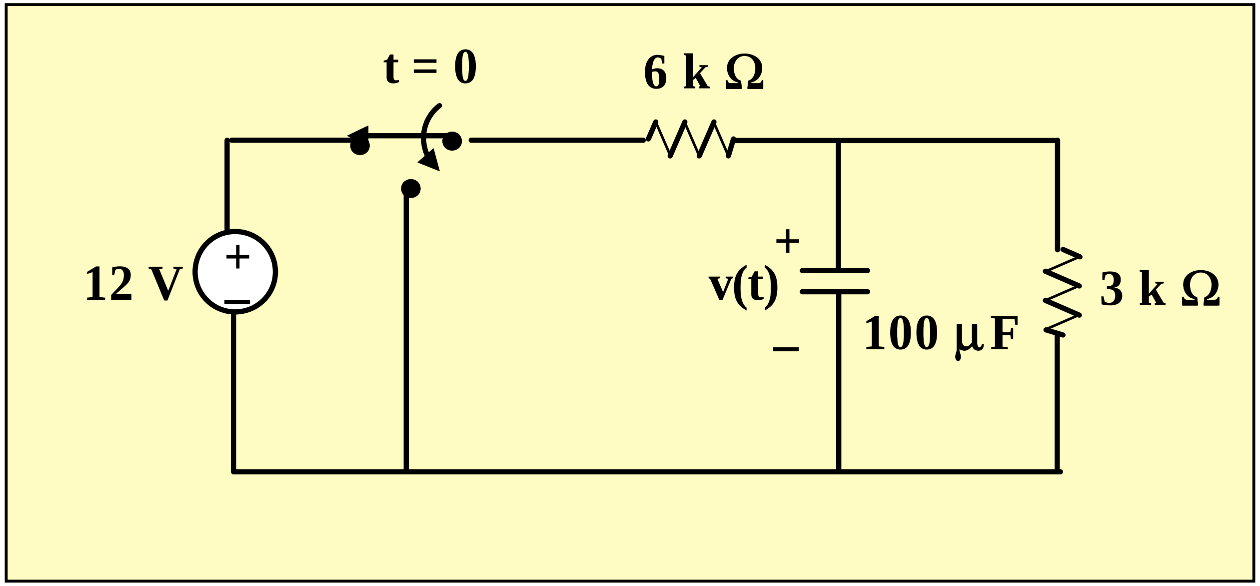



**Use Laplace transforms to find v(t) for t > 0.**

# Inverse Laplace Transforms

**Circuit theory problem:**

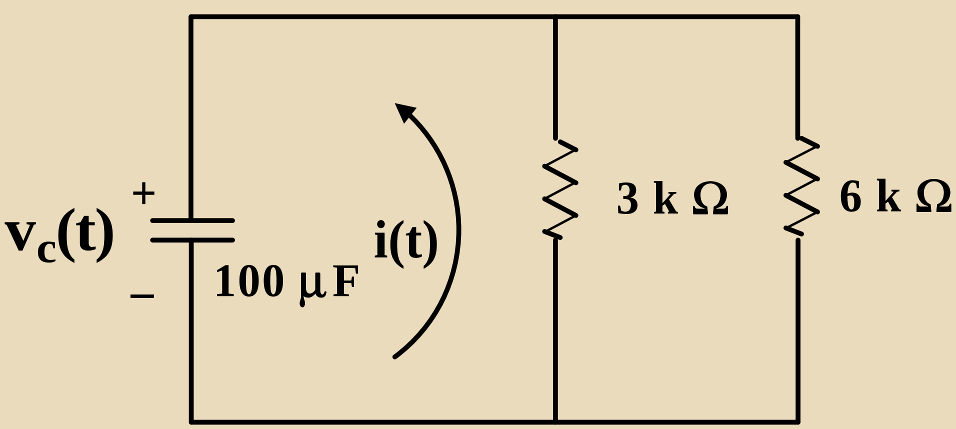
We see from the circuit,




$$v(0) = 12x\frac{3}{9} = 4\,volts$$

# Inverse Laplace Transforms

**Circuit theory problem:**




$$RC\frac{dv_c(t)}{dt} + v_c(t) = 0$$

$$\frac{dv_c(t)}{dt} + \frac{v_c(t)}{RC} = 0$$

$$\frac{dv_c(t)}{dt} + 5v_c(t) = 0$$

Take the Laplace transform of this equations including the initial conditions on $v_c(t)$

# Inverse Laplace Transforms

**Circuit theory problem:**

$$\frac{dv_c(t)}{dt} + 5v_c(t) = 0$$

$$sV_c(s) - 4 + 5V_c(s) = 0$$

$$V_c(s) = \frac{4}{s+5}$$

$$v_c(t) = 4e^{-5t}u(t)$$

# LAPLACE TRANSFORM

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# Definition

- The Laplace transform is a linear operator that switched a function f(t) to F(s).
- Specifically: $F(s) = \mathcal{L}\left\{f(t)\right\} = \int_{0^-}^{\infty} e^{-st} f(t)\, dt.$

  where: $s = \sigma + i\omega.$
- Go from time argument with real input to a complex angular frequency input which is complex.

# Restrictions

- There are two governing factors that determine whether Laplace transforms can be used:
  - $f(t)$ must be at least piecewise continuous for $t \geq 0$
  - $|f(t)| \leq Me^{\gamma t}$ where M and $\gamma$ are constants

# Continuity

- Since the general form of the Laplace transform is:

$$F(s) = \mathcal{L}\left\{f(t)\right\} = \int_{0^-}^{\infty} e^{-st} f(t)\,dt.$$

  it makes sense that f(t) must be at least piecewise continuous for $t \geq 0$.
- If f(t) were very nasty, the integral would not be computable.

# Boundedness

- This criterion also follows directly from the general definition:

$$F(s) = \mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} e^{-st} f(t)\, dt.$$

- If f(t) is not bounded by $Me^{\gamma t}$ then the integral will not converge.

# Laplace Transform Theory

- General Theory

$$F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt = \lim_{\tau\to\infty} \int_0^\tau e^{-st} f(t)dt$$

- Example

$$f(t) \equiv 1$$

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} 1 dt = \lim_{\tau\to\infty} \left( \frac{e^{-st}}{-s} \bigg|_0^\tau \right)$$

$$= \lim_{\tau\to\infty} \left( \frac{e^{-s\tau}}{-s} + \frac{1}{s} \right) = \frac{1}{s}$$

- Convergence

$$f(t) \equiv e^{t^2}$$

$$\mathcal{L}(f(t)) = \lim_{\tau\to\infty} \int_0^\tau e^{-st} e^{t^2} dt = \lim_{\tau\to\infty} \int_0^\tau e^{t^2 - st} dt = \infty$$

# Laplace Transforms

- Some Laplace Transforms
- Wide variety of function can be transformed

- Inverse Transform

$$\mathcal{L}^{-1}\big(F(s)\big) = f(t)$$

- Often requires partial fractions or other manipulation to find a form that is easy to apply the inverse

**TABLE 6.2.1** Elementary Laplace Transforms

| | $f(t) = \mathcal{L}^{-1}\{F(s)\}$ | $F(s) = \mathcal{L}\{f(t)\}$ |
|---|---|---|
| 1. | $1$ | $\frac{1}{s}, \quad s > 0$ |
| 2. | $e^{at}$ | $\frac{1}{s-a}, \quad s > a$ |
| 3. | $t^n, \quad n =$ positive integer | $\frac{n!}{s^{n+1}}, \quad s > 0$ |
| 4. | $t^p, \quad p > -1$ | $\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$ |
| 5. | $\sin at$ | $\frac{a}{s^2+a^2}, \quad s > 0$ |
| 6. | $\cos at$ | $\frac{s}{s^2+a^2}, \quad s > 0$ |
| 7. | $\sinh at$ | $\frac{a}{s^2-a^2}, \quad s > \lvert a \rvert$ |
| 8. | $\cosh at$ | $\frac{s}{s^2-a^2}, \quad s > \lvert a \rvert$ |
| 9. | $e^{at}\sin bt$ | $\frac{b}{(s-a)^2+b^2}, \quad s > a$ |
| 10. | $e^{at}\cos bt$ | $\frac{s-a}{(s-a)^2+b^2}, \quad s > a$ |
| 11. | $t^n e^{at}, \quad n =$ positive integer | $\frac{n!}{(s-a)^{n+1}}, \quad s > a$ |
| 12. | $u_c(t)$ | $\frac{e^{-cs}}{s}, \quad s > 0$ |
| 13. | $u_c(t)f(t-c)$ | $e^{-cs}F(s)$ |
| 14. | $e^{ct}f(t)$ | $F(s-c)$ |
| 15. | $f(ct)$ | $\frac{1}{c}F\left(\frac{s}{c}\right), \quad c > 0$ |
| 16. | $\int_0^t f(t-\tau)g(\tau)\,d\tau$ | $F(s)G(s)$ |
| 17. | $\delta(t-c)$ | $e^{-cs}$ |
| 18. | $f^{(n)}(t)$ | $s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$ |
| 19. | $(-t)^n f(t)$ | $F^{(n)}(s)$ |

# Laplace Transform for ODEs

- Equation with initial conditions

$$\frac{d^2y}{dt^2} + y = 1, \qquad y(0) = y'(0) = 0$$

- Laplace transform is linear

$$\mathcal{L}(y'') + \mathcal{L}(y) = \mathcal{L}(1)$$

- Apply derivative formula

$$s^2\mathcal{L}(y) - sy(0) - y'(0) + \mathcal{L}(y) = \frac{1}{s}$$

- Rearrange

$$\mathcal{L}(y) = \frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

- Take the inverse

$$y = 1 - \cos t$$

# Laplace Transform in PDEs

Laplace transform in two variables (always taken with respect to time variable, t):

$$\mathcal{L}\{u(x,t)\} = U(x,s) = \int_0^{\infty} e^{-st} \frac{du}{dt} dt$$

Inverse laplace of a 2 dimensional PDE:

$$\mathcal{L}^{-1}\{U(x,s)\} = u(x,t)$$

Can be used for any dimension PDE:

$$\mathcal{L}\{u(x,y,z,t)\} = U(x,y,z,s)$$

The Transform reduces dimension by “1”:

- •ODEs reduce to algebraic equations
- •PDEs reduce to either an ODE (if original equation dimension 2) or another PDE (if original equation dimension >2)

Consider the case where:

$u_x+u_t=t$ with $u(x,0)=0$ and $u(0,t)=t^2$ and

Taking the Laplace of the initial equation leaves $U_x + U=1/s^2$ (note that the partials with respect to "x" do not disappear) with boundary condition $U(0,s)=2/s^3$

Solving this as an ODE of variable x, $U(x,s)=c(s)e^{-x} + 1/s^2$

Plugging in B.C., $2/s^3=c(s) + 1/s^2$ so $c(s)=2/s^3 - 1/s^2$

$U(x,s)=(2/s^3 - 1/s^2)\ e^{-x} + 1/s^2$

Now, we can use the inverse Laplace Transform with respect to s to find

$u(x,t)=t^2e^{-x} - te^{-x} + t$

# Example Solutions

# Diffusion Equation

$u_t = ku_{xx}$ in $(0,l)$
Initial Conditions:
$u(0,t) = u(l,t) = 1$, $u(x,0) = 1 + \sin(\pi x/l)$

Using af(t) + bg(t) → aF(s) + bG(s)
and df/dt → sF(s) – f(0)
and noting that the partials with respect to x commute with the transforms with respect to t, the Laplace transform U(x,s) satisfies
$sU(x,s) - u(x,0) = kU_{xx}(x,s)$

With $e^{at} \rightarrow 1/(s-a)$ and a=0,
the boundary conditions become U(0,s) = U(l,s) = 1/s.

So we have an ODE in the variable x together with some boundary conditions. The solution is then:
$U(x,s) = 1/s + (1/(s+k\pi^2/l^2))\sin(\pi x/l)$
Therefore, when we invert the transform, using the Laplace table:
$u(x,t) = 1 + e^{-k\pi^2 t/l^2}\sin(\pi x/l)$

# Wave Equation

$u_{tt} = c^2u_{xx}$ in $0 < x < \infty$

Initial Conditions:

$u(0,t) = f(t)$, $u(x,0) = u_t(x,0) = 0$

For x → ∞, we assume that u(x,t) → 0. Because the initial conditions vanish, the Laplace transform satisfies

$s^2U = c^2U_{xx}$

U(0,s) = F(s)

Solving this ODE, we get

$U(x,s) = a(s)e^{-sx/c} + b(s)e^{sx/c}$

Where a(s) and b(s) are to be determined.

From the assumed property of u, we expect that U(x,s) → 0 as x → ∞.

Therefore, b(s) = 0. Hence, $U(x,s) = F(s)\, e^{-sx/c}$. Now we use

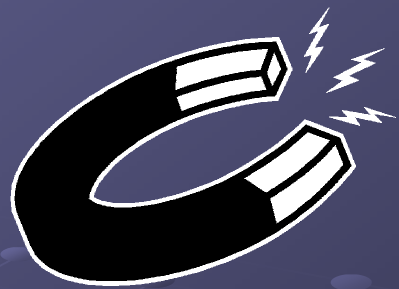
$H(t-b)f(t-b) \rightarrow e^{-bs}F(s)$

To get

u(x,t) = H(t – x/c)f(t – x/c).

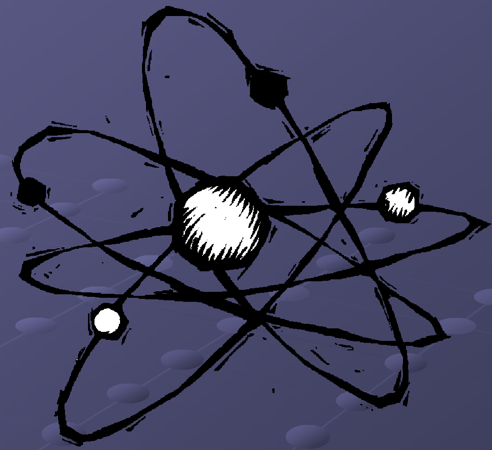
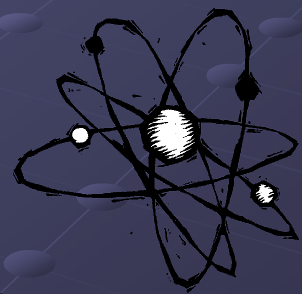
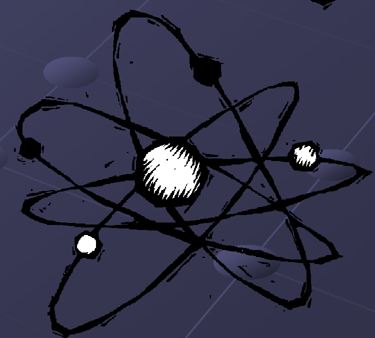
# Real-Life Applications

- Semiconductor mobility
- Call completion in wireless networks
- Vehicle vibrations on compressed rails
- Behavior of magnetic and electric fields above the atmosphere



# Ex. Semiconductor Mobility

- Motivation
  - semiconductors are commonly made with superlattices having layers of differing compositions
  - need to determine properties of carriers in each layer
    - concentration of electrons and holes
    - mobility of electrons and holes
  - conductivity tensor can be related to Laplace transform of electron and hole densities

# PARTIAL DIFFERENTIATION.

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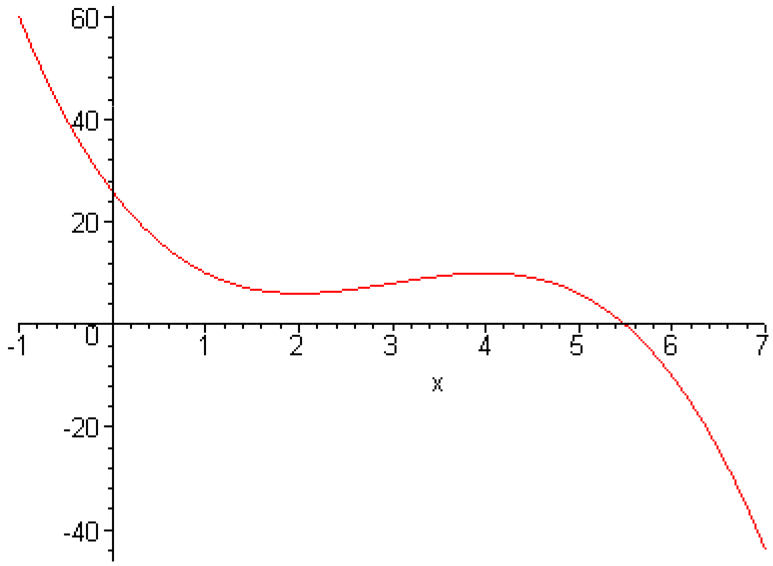
- How to get first order partial derivatives
- What is partial differentiation
- How to get second order partial derivatives

## Graphs and derivatives for function of one variable

**Example**

$f(x) = -x^3 + 9x^2 - 24x + 26$ OR $y = -x^3 + 9x^2 - 24x + 26$

**Graph:** a curve in 2 dimensions



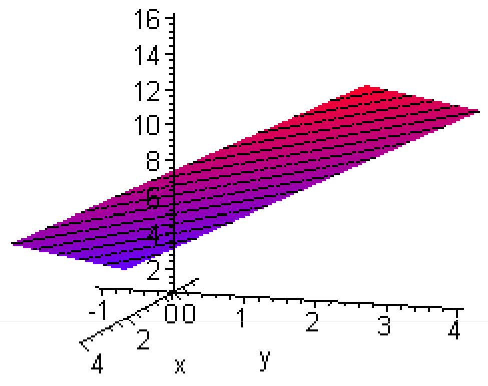

**Ordinary differentiation:**

$$\frac{dy}{dx} = -3x^2 + 18x - 24$$

$$\frac{d^2y}{dx^2} = -6x + 18$$

# Functions of several variables

**Example** $f(x,y) = x + 2y + 4$ OR $z = x + 2y + 4$




**Graph:** a surface in 3 dimensions

**Partial differentiation**

To differentiate *z* partially with respect to *x*

..treat *y* as a constant

..then differentiate w.r.t. *x*: --see next slide

N.B. $\partial$ ..denotes partial differentiation

d ..denotes ordinary differentiation

$\frac{\partial z}{\partial x}$ or $z_x$ ..denote the partial derivative of z wrt x

## Determine the first partial derivative of *z* w.r.t *x*: $\frac{\partial z}{\partial x}$ or $z_x$

Worked Example 7.2(a)

$$z = x + 2y + 4$$

..treat *y* as a constant

$$z = x + 2[y] + 4$$

..then differentiate *z* w.r.t. *x*:

$$\frac{\partial z}{\partial x} = 1 + 0 + 0 = 1$$

..since derivatives of constant terms (2*y* and 4) are zero

## Determine the first partial derivative of $z$ w.r.t $y$: or $z_y$ $\frac{\partial z}{\partial y}$

Worked Example 7.2(b)

$$z = x + 2y + 4$$

..treat $x$ as a constant

$$z = [x] + 2y + 4$$

..then differentiate w.r.t. $y$:

$$\frac{\partial z}{\partial y} = 0 + 2(1) + 0 = 2$$

..since derivatives of constant terms ($x$ and 4) are zero

# Worked Example 7.3

Find the first-order partial derivatives for each of the following functions.

(a) $z = 2x^2 + 3xy + 5$

(b) $Q = 10L^{0.7}K^{0.3}$

(c) $U = x^2 y^5$

## Determine the first partial derivative of *z* w.r.t *x*: $\frac{\partial z}{\partial x}$ or $z_x$

**Worked Example 7.3(a)**

$$z = 2x^2 + 3xy + 5$$

..treat *y* as a constant

$$z = 2x^2 + 3x[y] + 5$$

..then differentiate w.r.t. *x*:

$$\frac{\partial z}{\partial x} = 2(2x) + 3(1)y$$

$$\frac{\partial z}{\partial x} = 4x + 3y$$

## Determine the first partial derivative of *z* w.r.t *y*: $\frac{\partial z}{\partial y}$ or $z_y$

**Worked Example 7.3(a)**

$$z = 2x^2 + 3xy + 5$$

..treat *x* as a constant

$$z = 2[x^2] + 3[x]y + 5$$

..then differentiate *z* w.r.t. *y*:

$$\frac{\partial z}{\partial y} = 0 + 3x(1) + 0$$

$$\frac{\partial z}{\partial y} = 3x$$

Determine the first partial derivative of *Q* wrt *L*: $\frac{\partial Q}{\partial L}$ or $Q_L$

**Worked Example 7.3(b)** $Q = 10L^{0.7}K^{0.3}$

..treat *K* as a constant $Q = 10L^{0.7}[K^{0.3}]$

..then differentiate *Q* w.r.t. *L*: $\frac{\partial Q}{\partial L} = 10\left(0.7L^{0.7-1}\right)K^{0.3}$

$$\frac{\partial Q}{\partial L} = 7L^{-0.3}K^{0.3}$$

$\frac{\partial Q}{\partial K}$ or $Q_K$ the first partial derivative of *Q* wrt *K:*

**Worked Example 7.3(b)**

$$Q = 10L^{0.7}K^{0.3}$$

..treat *L* as a constant

$$Q = 10[L^{0.7}]K^{0.3}$$

..then differentiate *Q* w.r.t. *K:*

$$\frac{\partial Q}{\partial K} = 10L^{0.7}\left(0.3K^{0.3-1}\right)$$

$$\frac{\partial Q}{\partial K} = 3L^{0.7}K^{-0.7}$$

## $\frac{\partial U}{\partial x}$ the first partial derivative of *U* wrt *x*:

**Worked Example 7.3(c)**

$$U = x^2 y^5$$

..treat *y* as a constant

$$U = x^2[y^5]$$

..then differentiate *U* w.r.t. *x*:

$$\frac{\partial U}{\partial x} = (2x)y^5$$

$$\frac{\partial U}{\partial x} = 2xy^5$$

$\frac{\partial U}{\partial y}$ the first partial derivative of *U* wrt *y*:

**Worked Example 7.3(c)**

$$U = x^2 y^5$$

..treat *x* as a constant

$$U = [x^2]y^5$$

..then differentiate *U* w.r.t. *y*:

$$\frac{\partial U}{\partial y} = x^2(5y^4)$$

$$\frac{\partial U}{\partial y} = 5x^2y^4$$

# Worked Example 7.4

Find the second order partial derivatives for each of the following functions.

(a) $z = 2x^2 + 3xy + 5$

(b) $Q = 10L^{0.7}K^{0.3}$

(c) $U = x^2 y^5$

## $\frac{\partial^2 z}{\partial x^2}$ or $z_{xx}$ : second partial derivative of *z* wrt *x*:

**Worked Example 7.4(a)**

$$z = 2x^2 + 3xy + 5$$

..first partial derivative $z_x$

*see Worked Example 7.3*

$$\frac{\partial z}{\partial x} = 4x + 3y$$

..treat *y* as a constant

$$\frac{\partial z}{\partial x} = 4x + 3[y]$$

..differentiate $\frac{\partial z}{\partial x}$ w.r.t $x$

$$\frac{\partial^2 z}{\partial x^2} = 4(1) = 4 + 0$$

## $\frac{\partial^2 z}{\partial y^2}$ the second partial derivative of *z* wrt *y*:

**Worked Example 7.4(a)**

$$z = 2x^2 + 3xy + 5$$

..first partial derivative $z_y$

$$\frac{\partial z}{\partial y} = 3x$$

see Worked Example 7.3

..treat *x* as a constant

$$\frac{\partial z}{\partial y} = 3[x]$$

..differentiate $\frac{\partial z}{\partial y}$ w.r.t. *y*:

$$\frac{\partial^2 z}{\partial y^2} = 0$$

$$\frac{\partial^2 z}{\partial y \partial x}$$ the second partial derivative of *z* wrt *x* and *y*:

**Worked Example 7.4(a)**

$$z = 2x^2 + 3xy + 5$$

..first partial derivative $z_x$

$$\frac{\partial z}{\partial x} = 4x + 3y$$

*see Worked Example 7.3*

..treat *x* as a constant

$$\frac{\partial z}{\partial x} = 4[x] + 3y$$

..differentiate $\frac{\partial z}{\partial x}$ w.r.t. *y*: $\frac{\partial^2 z}{\partial y \partial x} = 0 + 3(1) = 3$

## $\frac{\partial^2 Q}{\partial L^2}$ or $Q_{LL}$: second partial derivative of $Q$ wrt $L$

**Worked Example 7.4(b)**

$$Q = 10L^{0.7}K^{0.3}$$

..first partial derivative: $Q_L$

$$\frac{\partial Q}{\partial L} = 7L^{-0.3}K^{0.3}$$

..treat $K$ as a constant

$$\frac{\partial Q}{\partial L} = 7L^{-0.3}[K^{0.3}]$$

..then differentiate $Q_L$ w.r.t. $L$:

$$\frac{\partial^2 Q}{\partial L^2} = 7(-0.3L^{-0.3-1})K^{0.3}$$

$$\frac{\partial^2 Q}{\partial L^2} = -2.1L^{-1.3}K^{0.3}$$

$$\frac{\partial^2 Q}{\partial K^2}$$ : the second partial derivative of *Q* wrt *K*

**Worked Example 7.4(b)**

$$Q = 10L^{0.7}K^{0.3}$$

..first partial derivative $Q_K$

$$\frac{\partial Q}{\partial K} = 3L^{0.7}K^{-0.7}$$

..treat $L$ as a constant

$$\frac{\partial Q}{\partial K} = 3[L^{0.7}]K^{-0.7}$$

..differentiate $Q_K$ w.r.t. $K$:

$$\frac{\partial^2 Q}{\partial K^2} = 3L^{0.7}\left(-0.7K^{-0.7-1}\right)$$

$$\frac{\partial^2 Q}{\partial K^2} = -2.1L^{0.7}K^{-1.7}$$

$$\frac{\partial^2 Q}{\partial K\, \partial L}$$ the second 'mixed' partial derivative of $Q$

**Worked Example 7.4(b)**

$$Q = 10L^{0.7}K^{0.3}$$

..first partial derivative: $Q_L$

$$\frac{\partial Q}{\partial L} = 7L^{-0.3}K^{0.3}$$

..treat $L$ as a constant

$$\frac{\partial Q}{\partial L} = 7[L^{-0.3}]K^{0.3}$$

..differentiate $Q_L$ w.r.t. $K$:

$$\frac{\partial^2 Q}{\partial K\, \partial L} = 7L^{-0.3}\left(0.3K^{0.3-1}\right)$$

$$\frac{\partial^2 Q}{\partial K\, \partial L} = 2.1L^{-1.3}K^{-0.7}$$

$$\frac{\partial^2 U}{\partial x^2}$$ the second partial derivative: $U_{xx}$

**Worked Example 7.4(c)**

$$U = x^2 y^5$$

..first partial derivative: $U_x$

$$\frac{\partial U}{\partial x} = 2xy^5$$

..treat $y$ as a constant

$$\frac{\partial U}{\partial x} = 2x[y^5]$$

..then differentiate $U_x$ w.r.t. $x$:

$$\frac{\partial^2 U}{\partial x^2} = 2(1)y^5 = 2y^5$$

$$\frac{\partial^2 U}{\partial y^2}$$ the second partial derivative: $U_{yy}$

**Worked Example 7.4(c)** $U = x^2 y^5$

..first partial derivative: $U_y$ $$\frac{\partial U}{\partial y} = 5x^2 y^4$$

..treat *x* as a constant $$\frac{\partial U}{\partial y} = 5[x^2]y^4$$

..then differentiate $U_y$ w.r.t. *y*: $$\frac{\partial^2 U}{\partial y^2} = 5x^2(4y^3) = 20x^2y^3$$

$$\frac{\partial^2 U}{\partial y \partial x}$$ the second mixed derivative: $U_{xy}$

**Worked Example 7.4(c)**

$$U = x^2 y^5$$

..first partial derivative: $U_x$

$$\frac{\partial U}{\partial x} = 2xy^5$$

..treat $x$ as a constant

$$\frac{\partial U}{\partial x} = 2[x]y^5$$

..then differentiate $U_x$ w.r.t. $y$:

$$\frac{\partial^2 U}{\partial y \partial x} = 2x(5y^4) = 10xy^4$$

# The Real Number System

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**Words to know….**

- **Naturals** – natural **counting** numbers **{ 1, 2, 3, ….}**
- **Wholes** – natural counting numbers and **zero** **{ 0,1, 2, 3….}**
- **Integers** – **Positive or negative** natural numbers or zero { -2, -1, 0, 1, 2,…}

**Words to know….**

**Rational Number** – any number which can be **written as a fraction**.

**Irrational Number** –

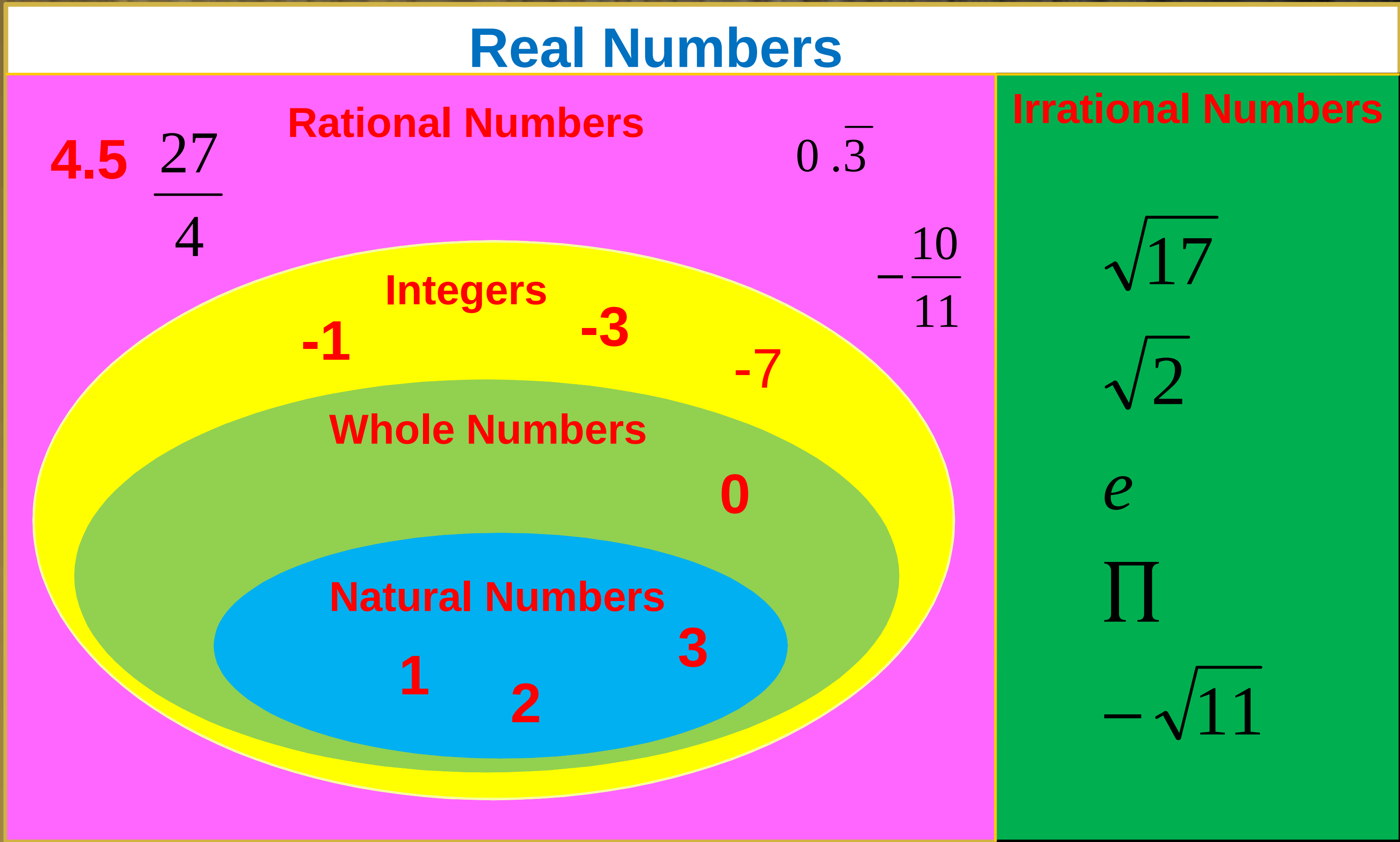
Any decimal number which can't be written as a fraction.

A non-terminating and non-repeating decimal.

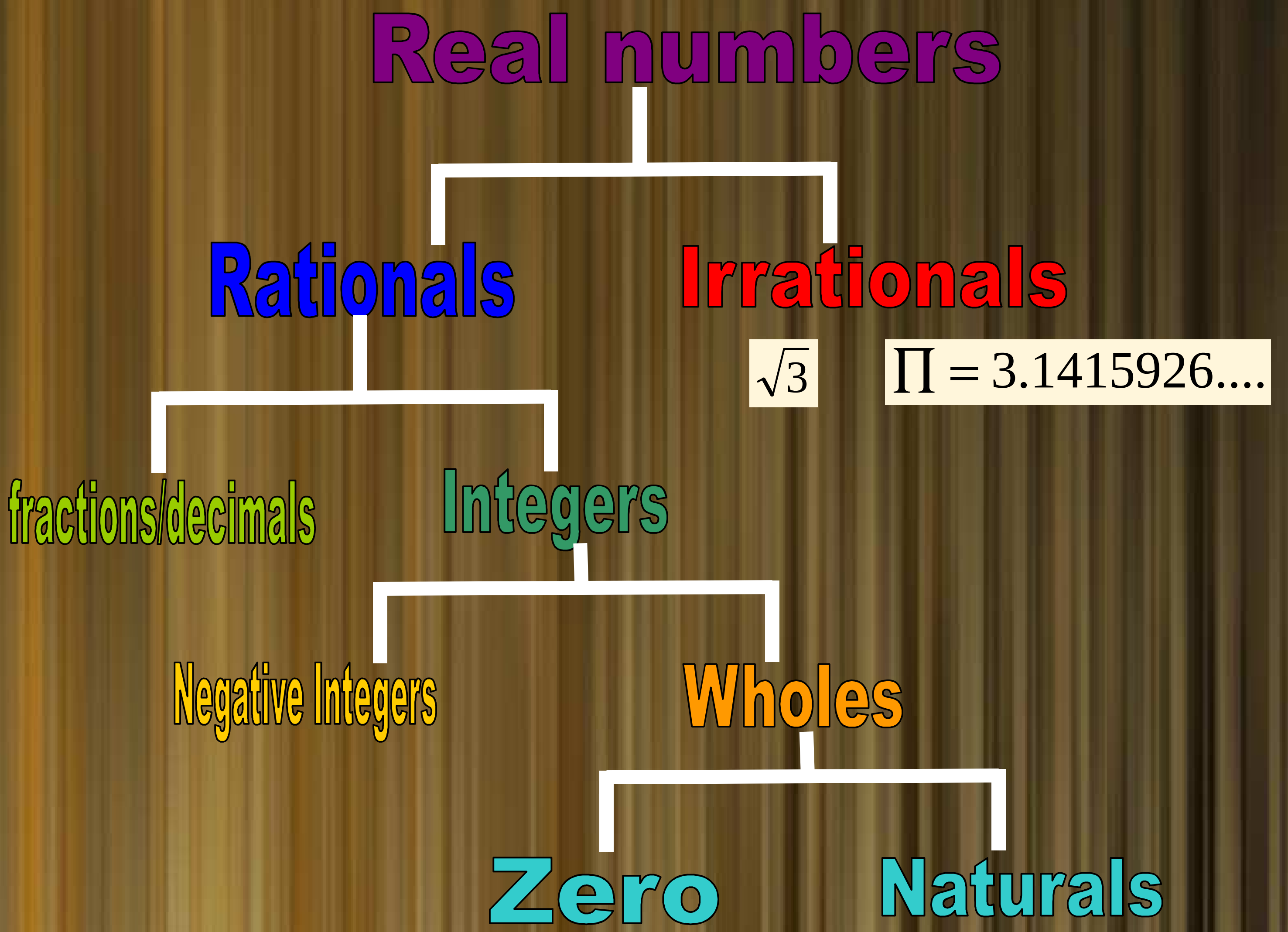
Example –

$\prod = 3.1415926....$

» **Real** – **rational** numbers and **irrational** numbers.


Real Numbers
Rational Numbers
4.5
$\frac{27}{4}$
$0.\overline{3}$
$-\frac{10}{11}$
Integers
-1
-3
-7
Whole Numbers
0
Natural Numbers
1
2
3
Irrational Numbers
$\sqrt{17}$
$\sqrt{2}$
$e$
$\Pi$
$-\sqrt{11}$

# Sets of Numbers – (using a tree map)



# Let's practice

Directions: Identify each number below as natural, whole, integer, rational, irrational, or real. More than one answer can apply.

1. $\frac{7}{8}$ — **Rational, real**

2. $0$ — **Whole, integer, rational, real**

3. $-9$ — **integer, rational, real**

4. $-\frac{4}{5}$ — **rational, real**

# Let's practice

Directions: Identify each number below as natural, whole, integer, rational, irrational, or real. More than one answer can apply.

5. $\sqrt{6}$ — irrational, real

7. $\sqrt{16}$ — Natural, Whole, integer, rational, real

6. $8$ — Natural, Whole, integer, rational, real

8. $-\sqrt{25}$ — integer, rational, real

# Use $<$, $>$, or $=$ to compare

1. $\frac{2}{3} = .\overline{6}$
2. $\frac{5}{8} < .65$
3. $\frac{5}{9} > .5$
4. $\frac{3}{10} < \frac{1}{3}$
5. $\frac{12}{12} = 1$
6. $\Pi > \frac{21}{7}$
7. $\frac{7}{6} < \frac{6}{5}$
8. $\frac{5}{20} = .25$

# Lesson Quiz

Write all classifications that apply to each number.

1. $\sqrt{2}$ real, irrational

2. $-\frac{\sqrt{16}}{2}$ real, integer, rational

3. $\frac{\sqrt{25}}{0}$

not a real number

4. $\sqrt{4} \bullet \sqrt{9}$

rational

# SEQUENCES AND SUMMATIONS

PROF. BELOTE S. V.
DEPT. OF MATHEMATICS

# Definitions

- Sequence: an ordered list of elements
  - Like a set, but:
    - Elements can be duplicated
    - Elements are ordered

# Sequences

- A sequence is a function from a subset of **Z** to a set S
  - Usually from the positive or non-negative ints
  - $a_n$ is the image of $n$

- $a_n$ is a term in the sequence
- $\{a_n\}$ means the entire sequence
  - The same notation as sets!

# Sequence examples

- $a_n = 3n$
  - The terms in the sequence are $a_1$, $a_2$, $a_3$, …
  - The sequence $\{a_n\}$ is { 3, 6, 9, 12, … }
- $b_n = 2^n$
  - The terms in the sequence are $b_1$, $b_2$, $b_3$, …
  - The sequence $\{b_n\}$ is { 2, 4, 8, 16, 32, … }
- Note that sequences are indexed from 1
  - Not in all other textbooks, though!

# Geometric vs. arithmetic sequences

- The difference is in how they grow

- Arithmetic sequences increase by a constant *amount*
  - $a_n = 3n$
  - The sequence $\{a_n\}$ is { 3, 6, 9, 12, … }
  - Each number is 3 more than the last
  - Of the form: $f(x) = dx + a$

- Geometric sequences increase by a constant *factor*
  - $b_n = 2^n$
  - The sequence $\{b_n\}$ is { 2, 4, 8, 16, 32, … }
  - Each number is twice the previous
  - Of the form: $f(x) = ar^x$

# Fibonacci sequence

- Sequences can be neither geometric or arithmetic
  - $F_n = F_{n-1} + F_{n-2}$, where the first two terms are 1
    - Alternative, $F(n) = F(n-1) + F(n-2)$
  - Each term is the sum of the previous two terms
  - Sequence: { 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, … }
  - This is the Fibonacci sequence
  - Full formula: $$F(n) = \frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{\sqrt{5}\cdot 2^n}$$

# Fibonacci sequence

- As the terms increase, the ratio between successive terms approaches 1.618

$$\lim_{n\to\infty}\frac{F(n+1)}{F(n)} = \phi = \frac{\sqrt{5}+1}{2} = 1.618933989$$

- This is called the “golden ratio”
  - Ratio of human leg length to arm length
  - Ratio of successive layers in a conch shell

- Reference: http://en.wikipedia.org/wiki/Golden_ratio

# Determining the sequence formula

- Given values in a sequence, how do you determine the formula?

- Steps to consider:
  - Is it an arithmetic progression (each term a constant amount from the last)?
  - Is it a geometric progression (each term a factor of the previous term)?
  - Does the sequence it repeat (or cycle)?
  - Does the sequence combine previous terms?
  - Are there runs of the same value?

# Determining the sequence formula

- Rosen, question 9 (page 236)

a) 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, …
   - The sequence alternates 1's and 0's, increasing the number of 1's and 0's each time

b) 1, 2, 2, 3, 4, 4, 5, 6, 6, 7, 8, 8, …
   - This sequence increases by one, but repeats all even numbers once

c) 1, 0, 2, 0, 4, 0, 8, 0, 16, 0, …
   - The non-0 numbers are a geometric sequence ($2^n$) interspersed with zeros

d) 3, 6, 12, 24, 48, 96, 192, …
   - Each term is twice the previous: geometric progression
   - $a_n = 3*2^{n-1}$

# Determining the sequence formula

e) 15, 8, 1, -6, -13, -20, -27, …
- Each term is 7 less than the previous term
- $a_n = 22 - 7n$

f) 3, 5, 8, 12, 17, 23, 30, 38, 47, …
- The difference between successive terms increases by one each time
- $a_1 = 3$, $a_n = a_{n-1} + n$
- $a_n = n(n+1)/2 + 2$

g) 2, 16, 54, 128, 250, 432, 686, …
- Each term is twice the cube of *n*
- $a_n = 2*n^3$

h) 2, 3, 7, 25, 121, 721, 5041, 40321
- Each successive term is about *n* times the previous
- $a_n = n! + 1$
- My solution: $a_n = a_{n-1} * n - n + 1$

# Useful sequences

- $n^2$ = 1, 4, 9, 16, 25, 36, …
- $n^3$ = 1, 8, 27, 64, 125, 216, …
- $n^4$ = 1, 16, 81, 256, 625, 1296, …
- $2^n$ = 2, 4, 8, 16, 32, 64, …
- $3^n$ = 3, 9, 27, 81, 243, 729, …
- $n!$ = 1, 2, 6, 24, 120, 720, …

- Listed in Table 1, page 228 of Rosen

# Evaluating sequences

- Rosen, question 13, page 3.2

$$\sum_{k=1}^{5}(k+1)$$

- 2 + 3 + 4 + 5 + 6 = 20

$$\sum_{j=0}^{4}(-2)^j$$

- $(-2)^0 + (-2)^1 + (-2)^2 + (-2)^3 + (-2)^4 = 11$

$$\sum_{i=1}^{10}3$$

- 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 = 30

$$\sum_{j=0}^{8}\left(2^{j+1}-2^j\right)$$

- $(2^1-2^0) + (2^2-2^1) + (2^3-2^2) + \ldots (2^{10}-2^9) = 511$
  - Note that each term (except the first and last) is cancelled by another term

# Evaluating sequences

- Rosen, question 14, page 3.2
  - S = { 1, 3, 5, 7 }

- What is $\Sigma_{j\in S}\, j$
  - 1 + 3 + 5 + 7 = 16
- What is $\Sigma_{j\in S}\, j^2$
  - $1^2 + 3^2 + 5^2 + 7^2 = 84$
- What is $\Sigma_{j\in S}\, (1/j)$
  - 1/1 + 1/3 + 1/5 + 1/7 = 176/105
- What is $\Sigma_{j\in S}\, 1$
  - 1 + 1 + 1 + 1 = 4

# Summation of a geometric series

- Sum of a geometric series:

$$\sum_{j=0}^{n} ar^{j} = \begin{cases} \dfrac{ar^{n+1} - a}{r-1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$$

- Example:

$$\sum_{j=0}^{10} 2^{n} = \frac{2^{10+1} - 1}{2-1} = \frac{2048-1}{1} = 2047$$

# Proof of last slide

- If $r$ = 1, then the sum is:

$$S = \sum_{j=0}^{n} a = (n+1)a$$

$$S = \sum_{j=0}^{n} ar^j$$

# Double summations

- Like a nested for loop

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij$$

- Is equivalent to:

```
int sum = 0;
for ( int i = 1; i <= 4; i++ )
    for ( int j = 1; j <= 3; j++ )
        sum += i*j;
```

# Useful summation formulae

- Well, only 1 really important one:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$